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SOME RESULTS ON LOCAL COHOMOLOGY MODULES WITH RESPECT TO A PAIR OF IDEALS

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Abstract

We introduce some results for the local cohomology modules with respect to a pair of ideals (I, J), which is a generalization of the concept of local cohomology modules with respect to an ideal I.

1 Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity.

Local cohomology was introduced by Grothendieck and many people have worked about the understanding of their structure, (non)-vanishing and finiteness properties. For example, Grothendieck's non-vanishing theorem is one of the important theorems in local cohomology. For more details about local cohomology modules, see [2].

In [5], Takahashi, Yoshino and Yoshizawa introduced the local cohomology modules with respect to a pair of ideals (I, J). For an R-module M, the (I, J)-torsion submodule of M is

$$\Gamma_{I,J}(M) = \{x \in M : I^n x \subset Jx \text{ for some positive integer } n\}.$$

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$\Gamma_{I,J}(\bullet)$ is a covariant functor from the category of R-modules to itself. The i-th local cohomology functor $\mathrm{H}^i_{I,J}(\bullet)$ with respect to (I,J) is defined to be the i-th right derived functor of $\Gamma_{I,J}(\bullet)$. When $J=0$, the $\mathrm{H}^i_{I,J}(\bullet)$ coincides with the usual local cohomology functor $\mathrm{H}^i_I(\bullet)$.

We observe also that there exists the theory of formal local cohomology modules, and some results for a such module are found in [6].

The organization of the paper is as follows.

In the Section 2, we put some basic properties of local cohomology modules with respect to a pair of ideals. And also, we put some results.

In the Section 3, we presented some results about the theory in question, with applications. We put some applications which involve the R-module $\mathrm{H}^i_{I,J}(M)$, with M an R-module finitely generated.

We finalize the paper, with a conclusion.

Here we use properties of commutative algebra and homological algebra for the development of the results (see [1] and [4]).

2 The results about local cohomology

In this section, we presented some results about the modules and submodules which involve the theory in question.

The next, we have the following definition.

Definition 2.1. Let M be an R-module and let I, J be two ideals of R. Following [5], we denote by $\Gamma_{I,J}(M)$ the set of all elements $x \in M$ such that $I^n x \subseteq Jx$, for some integer $n \in \mathbb{N}$.

Notice that an element $x \in M$ belongs to $\Gamma_{I,J}(M)$ if and only if

$$I^n \subseteq (0 :_R x) + J,$$

for some integer $n \in \mathbb{N}$.

Then, $\Gamma_{I,J}(\bullet)$, called the (I,J)-torsion functor becomes a covariant, R-linear and left exact functor from $\mathfrak{C}(R)$ to itself, according to [5].

It is clear that if $J=0$, then (I,J)-torsion functor $\Gamma_{I,J}(\bullet)$ coincides with I-torsion functor $\Gamma_I(\bullet)$, according to [2].

We have now the following definition.

Definition 2.2. For an integer $i \geq 0$ and for an R-module M, we refer to $\mathrm{H}^i_{I,J}(M)$, according to [5], as the i-th local cohomology of M relative to (I,J), which is defined by:

$$\mathrm{H}^i_{I,J}(M) := H^i(\Gamma_{I,J}(E^\bullet)),$$

where $E^\bullet$ denotes a (minimal) injective resolution of M.

Let $I_R = (0 :_R R/I)$ denote the annihilator of the R-module R/I. For each ideal K of R, let $\tilde{W}(K, J)$ denote the set of all ideals $\mathfrak{a}$ of R such that $K^n \subseteq \mathfrak{a} + J$, for some integer $n \in \mathbb{N}$. Then, $\tilde{W}(K, J)$ is a partially ordered set with the relation:

$$\mathfrak{a} \leq \mathfrak{b} \text{ iff } \mathfrak{b} \subseteq \mathfrak{a}.$$

Also, if $\mathfrak{a} \leq \mathfrak{b}$, then for each R-module M, we have that

$$\Gamma_{\mathfrak{a}}(M) \subseteq \Gamma_{\mathfrak{b}}(M).$$

So, for each R-module M the order relation on $\tilde{W}(K, J)$ and the inclusion maps make $\{\Gamma_{\mathfrak{a}}(M)\}_{\mathfrak{a} \in \tilde{W}(K,J)}$ into a direct system of R-modules.

Theorem 2.3. *We have that for an R-module M:*

$$\left(\mathrm{H}^i_{I,J}(M)\right)_{i \in \mathbb{N}_0} \cong \left(\varinjlim_{\mathfrak{a} \in \tilde{W}(I_R,J)} \mathrm{H}^i_{\mathfrak{a}}(M) \right)_{i \in \mathbb{N}_0} \cong \left(\varinjlim_{\mathfrak{a} \in \tilde{W}(I,J)} \mathrm{H}^i_{\mathfrak{a}}(M) \right)_{i \in \mathbb{N}_0},$$

as connected sequences of functors from $\mathfrak{C}(R)$ to itself.

Proof. We only prove the first isomorphism. That the first and third connected sequences are isomorphic can be proved in a similar way.

First we show that

$$\Gamma_{I,J}(M) = \varinjlim_{\mathfrak{a} \in \tilde{W}(I_R,J)} \Gamma_{\mathfrak{a}}(M).$$

To this end, note that the right hand is in fact

$$\bigcup_{\mathfrak{a} \in \tilde{W}(I_R,J)} \Gamma_{\mathfrak{a}}(M).$$

So, it is enough to show that

$$\Gamma_{I,J}(M) = \bigcup_{\mathfrak{a} \in \tilde{W}(I_R,J)} \Gamma_{\mathfrak{a}}(M).$$

To do this, let $f \in \Gamma_{I,J}(M)$. Then, there exists an integer $n > 0$ such that we have $I^n \subseteq (0 :_R f(R)) + J$. Since $(0 :_R R) \subseteq (0 :_R f(R))$, and R is a Noetherian ring, we have that

$$I_R^t \subseteq (0 :_R f(R)) + J,$$

for sufficiently large t. So setting $\mathfrak{a} = (0 :_R f(R))$, we have that $\mathfrak{a} \in \tilde{W}(I_R, J)$ and $f \in \Gamma_{\mathfrak{a}}(M)$.

Conversely, let

$$f \in \bigcup_{\mathfrak{a} \in \tilde{W}(I_R, J)} \Gamma_{\mathfrak{a}}(M).$$

Then, there exists an ideal $\mathfrak{a} \in \tilde{W}(I_R, J)$ with $f \in \Gamma_{\mathfrak{a}}(M)$. Thus, it follows that we have

$$I_R^m \subseteq \mathfrak{a} + J \quad \text{and} \quad \mathfrak{a}^n f(R) = 0,$$

for integers m, $n > 0$. Therefore, we have that

$$I^{mn} f(R) \subseteq J f(R) \;\; \text{and hence} \quad f \in \Gamma_{I,J}(M).$$

Now let

$$0 \to X \to Y \to Z \to 0,$$

be an exact sequence of R-modules. It implies the long exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{a}}(X) \to \mathrm{H}^0_{\mathfrak{a}}(Y) \to \mathrm{H}^0_{\mathfrak{a}}(Z) \to \mathrm{H}^1_{\mathfrak{a}}(X) \to \mathrm{H}^1_{\mathfrak{a}}(Y) \to \dots,$$

for each $\mathfrak{a} \in \tilde{W}(I_R, J)$.

Since taking the direct limit preserve exactness, we obtain the long exact sequence

$$0 \to \varinjlim_{\mathfrak{a} \in \tilde{W}(I_R,J)} \mathrm{H}^0_{\mathfrak{a}}(X) \to \varinjlim_{\mathfrak{a} \in \tilde{W}(I_R,J)} \mathrm{H}^0_{\mathfrak{a}}(Y) \to \varinjlim_{\mathfrak{a} \in \tilde{W}(I_R,J)} \mathrm{H}^0_{\mathfrak{a}}(Z) \to \varinjlim_{\mathfrak{a} \in \tilde{W}(I_R,J)} \mathrm{H}^1_{\mathfrak{a}}(X) \to \varinjlim_{\mathfrak{a} \in \tilde{W}(I_R,J)} \mathrm{H}^1_{\mathfrak{a}}(Y) \to \dots .$$

We have thus made

$$\left(\varinjlim_{\mathfrak{a} \in \tilde{W}(I_R,J)} \mathrm{H}^i_{\mathfrak{a}}(\bullet) \right)_{i \in \mathbb{N}_0},$$

into a connected sequence of functors. Since

$$\varinjlim_{\mathfrak{a} \in \tilde{W}(I_R,J)} \mathrm{H}^i_{\mathfrak{a}}(E) = 0,$$

for all $i > 0$, whenever E is an injective R-module, the claim it follows by [2, Theorem 1.3.5]. This finishes the proof. $\square$

Remark 2.4. Let $\{a_1, \dots, a_n\}$ be a generating set for the ideal $\mathfrak{a}$, and let $K^t_\bullet$ denote the Koszul complex of R with respect to $a_1^t, \dots, a_n^t$. Let $P_\bullet$ be a projective resolution for R. If $C^t_\bullet$ denotes the single complex associated to the double complex $K^t_\bullet \otimes_R P_\bullet$, then following [2] we have that

$$\mathrm{H}^i_{\mathfrak{a}}(M) \cong \varinjlim_{t \in \mathbb{N}} H^i(\mathrm{Hom}_R(C^t_\bullet, M)),$$

for all $i \in \mathbb{N}_0$.

Therefore, it follows that, for any exact sequence

$$0 \to X \to Y \to Z \to 0,$$

being R a Noetherian ring, we have the long exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{a}}(Z) \to \mathrm{H}^0_{\mathfrak{a}}(Y) \to \mathrm{H}^0_{\mathfrak{a}}(X) \to \dots,$$

of local cohomology modules.

In the same context, it follows the following result which involve the theory in question.

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Suppose that $J \neq R$. Then, we have that*

$$\mathrm{H}^i_{I,J}(M) = 0,$$

for all $i > \mathrm{pd}(R) + \dim_R(M/JM)$.

Proof. We consider in the proof $p = \mathrm{pd}(R)$. Thus, there exists a finitely generated free R-module F and a submodule L of F such that $\mathrm{pd}(L) = p - 1$, and that the sequence

$$0 \to L \to F \to R \to 0,$$

is exact. Using Remark 2.4 and the Theorem 2.3, this yields the long exact sequence

$$\mathrm{H}^{i-1}_{I,J}(L) \to \mathrm{H}^i_{I,J}(M) \to \mathrm{H}^i_{I,J}(F),$$

for all $i > 0$, and the result it follows. □

Remark 2.6. It should be mentioned that the functor $\mathrm{H}^i_{\mathfrak{a}}(\bullet)$ commutes with direct limits in the category of all R-modules. Also, any two direct limits commute.

3 Some results of applications

We presented now the following result.

Proposition 3.1. *Let $(R, \mathfrak{m})$ be a local ring and $d = \dim_R(R)$. Assume that M is a finitely generated R-module. Then for each ideal $\mathfrak{a}$ of R, we have that*

$$\mathrm{H}^i_{\mathfrak{a}}(M) = 0,$$

for all $i > d$.

Proof. See [3, Theorem 3.1]. □

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a local ring and $d = \dim_R(R)$. Then, for any R-module M (not necessary finitely generated), we have that*

$$\mathrm{H}^i_{I,J}(M) = 0,$$

for all $i > d$.

Proof. It follows by Proposition 3.1, and, Theorem 2.3. □

For a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, we denote by $E(R/\mathfrak{p})$ the injective envelope of the R-module $R/\mathfrak{p}$.

We conclude, then, the paper with a theorem that involves the previously presented concepts.

Theorem 3.3. *Let $\mathfrak{p} \in \mathrm{Spec}(R)$. If $\mathfrak{p} \in W(I_R, J)$, then we have that*

$$\Gamma_{I,J}(E(R/\mathfrak{p})) = E(R/\mathfrak{p}).$$

On the other hand if $\mathfrak{p} \notin W(I_R, J)$, then we have that

$$\Gamma_{I,J}(E(R/\mathfrak{p})) = 0.$$

Proof. If $\mathfrak{p} \in W(I_R, J)$, then we have that

$$\mathrm{Ass}_R(E(R/\mathfrak{p})) = \{\mathfrak{p}\} \subseteq W(I_R, J).$$

Therefore, we have that

$$\Gamma_{I,J}(E(R/\mathfrak{p})) = E(R/\mathfrak{p}),$$

by [5, Proposition 1.7].

Now let $\mathfrak{p} \notin W(I_R, J)$. Then, it follows that

$$\mathrm{Ass}_R(E(R/\mathfrak{p})) \bigcap W(I_R, J) \subseteq \{\mathfrak{p}\} \bigcap W(I_R, J) = \emptyset.$$

Therefore, by [5, Proposition 1.10], we have that

$$\Gamma_{I,J}(E(R/\mathfrak{p})) = 0.$$

This finishes the proof. □

We finished the article with the following conclusion.

4 Conclusion

In this article, we can to relate the theory of homological algebra to the theory of local cohomology modules. With the results of the article, we show the importance of local cohomology theory as a study tool within of the commutative algebra theory.

Moreover, by making this relationship, we get applications for the local cohomology module with respect to a pair of ideals in a general theory of modules.

5 Conflicting interests

The author claims that there are no conflicting interests.

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