

EXISTENCE OF WEAK SOLUTIONS FOR THE KIRCHHOFF-TYPE EQUATION WITH MIXED BOUNDARY CONDITIONS IN VARIABLE EXPONENT SOBOLEV SPACE

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Abstract

In this paper, we consider the Kirchhoff-type equation for a class of nonlinear operators containing $p(\cdot)$ -Laplacian and mean curvature operator with mixed boundary conditions. More precisely, we are concerned with the problem under the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of one, two and infinitely many nontrivial weak solutions of the equation according to the conditions on given functions.

Keywords.

1 Introduction

In this paper, we consider the following Kirchhoff-type equation

$$\begin{cases} -M\left(\int_{\Omega} A(y, \nabla u(y))dy\right) \operatorname{div} [\mathbf{a}(x, \nabla u(x))] = f(x, u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ M\left(\int_{\Omega} A(y, \nabla u(y))dy\right) \mathbf{n}(x) \cdot \mathbf{a}(x, \nabla u(x)) = g(x, u(x)) & \text{on } \Gamma_2. \end{cases} \quad (1.1)$$

Here Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with a Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint open subsets of } \Gamma \text{ such that } \overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset, \quad (1.2)$$

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Key words: Kirchhoff-type equation, $p(\cdot)$ -Laplacian, mean curvature operator, mixed boundary value problem, variable exponent Sobolev space.

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and the vector field $\mathbf{n}$ denotes the unit, outer, normal vector to Γ . The function $\mathbf{a}(x, \boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} A(x, \boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$ satisfying some structure conditions depending on an anisotropic exponent function $p(x)$. Then the operator $\operatorname{div}[\mathbf{a}(x, \nabla u(x))]$ is more general than the $p(\cdot)$ -Laplacian $\Delta_{p(x)} u(x) = \operatorname{div}[|\nabla u(x)|^{p(x)-2} \nabla u(x)]$ and the mean curvature operator $\operatorname{div}[(1 + |\nabla u(x)|^2)^{(p(x)-2)/2} \nabla u(x)]$. These generalities bring about difficulties and requires some conditions.

We impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2 . The given data $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : \Gamma_2 \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions satisfying some conditions.

The study of differential equations with $p(\cdot)$ -growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [40]), in electrorheological fluids (Diening [16], Halsey [25], Mihăilescu and Rădulescu [30], Růžička [33]).

Over the last two decades, when $M(t) \equiv 1$, there are many articles on the existence of weak solutions for the Dirichlet boundary condition, that is, in the case $\Gamma_2 = \emptyset$ in (1.1), (for example, see Mashiyev et al. [29], Duc and Vu [17], Wei and Chen [34], Yücedağ [38], Nápoli and Mariani [31]).

The Kirchhoff-type equation has been considered by many authors. The original Kirchhoff equation introduced by Kirchhoff [27] is as follows.

$$\rho \frac{\partial^2 u}{\partial s^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where s denotes the time, ρ is the mass density, ρ_0 is the initial tension, h is the area of the cross section, E is the Young modulus of the material and L is the length of the string. Such type of system is an extension of the classical D'Alembert wave equation, by considering the effects of the changes in the length of the string during the vibration. For some interesting results, see Arosio and Pannizi [10], Cavalcante et al. [11], Corrêa and Figueiredo [13], D'Ancona and Spagnolo [15], and He and Zou [26].

The stationary analogue of the Kirchhoff equation with the Dirichlet boundary condition takes the form

$$\begin{cases} - \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

where a and b are positive constants.

Since we can only find a few of papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1). See Aramaki [2, 5]. We are convinced of the reason for existence of this paper.

Dai and Hao [14] considered the problem (1.1) when $A(x, \xi) = \frac{1}{p(x)}|\xi|^{p(x)}$ and $\Gamma_2 = \emptyset$, and derived the existence of a nontrivial weak solution to (1.1). This paper is an extension of the article [14] to the case of mixed boundary value problem and of a class of operators containing the $p(\cdot)$ -Laplacian and the mean curvature operator. In Aramaki [4] the author treated the case where $A(x, \xi)$ is uniformly convex, which is different from the condition of this paper. In this paper, we give up this condition, however we assume that $\mathbf{a}(x, \xi)$ is uniformly monotone in the more natural sense. For example, in the papers [29] and [38], they considered $p(\cdot)$ -Laplacian in only the case $p(x) \geq 2$ in $\bar{\Omega}$.

The purpose of the paper is that we treat not only $p(\cdot)$ -Laplacian, but also the mean curvature operator in the case $p(x) > 1$ in $\bar{\Omega}$. To overcome this, we are saved by the grace of Proposition 3.7 later which is firstly derived by the author Aramaki [8, Proposition 3.5]. Thanks to this proposition, we can handle not only the case $p(x) \geq 2$ but also the case $p(x) > 1$. We derive that there exist one, two and infinitely many nontrivial weak solutions. We use the standard Mountain-Pass Theorem, Ekeland variational principle and the Symmetric Mountain-Pass Theorem, respectively.

The paper is organized as follows. In Section 2, we recall some well-known results on variable exponent Lebesgue-Sobolev spaces. In Section 3, we give the assumptions to the main theorems (Theorem 4.3, 4.5 and 4.6). In Section 4, we state the main theorems on the existence of at least one, two and infinitely many nontrivial weak solutions according to the hypotheses on given functions f and g . The proofs of these main theorems are given in Section 5.

2 Preliminaries

Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with a $C^{0,1}$ -boundary Γ and Ω is locally on the same side of Γ . Moreover, we assume that Γ satisfies (1.2).

In the present paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B , we denote B^N by the boldface character $\mathbf{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\mathbf{a} = (a_1, \dots, a_N)$ and $\mathbf{b} = (b_1, \dots, b_N)$ in $\mathbb{R}^N$ by $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^N a_i b_i$ and $|\mathbf{a}| = (\mathbf{a} \cdot \mathbf{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*, B}$.

We recall some well-known results on variable exponent Lebesgue and Sobolev spaces. See Fan and Zhang [21], Kováčik and Rákosník [28] and references therein for more detail. Furthermore, we consider some new properties on variable exponent Lebesgue space. Define $C(\bar{\Omega}) = \{p; p \text{ is a continuous function on } \bar{\Omega}\}$, and for any $p \in C(\bar{\Omega})$, put

$$p^+ = p^+(\Omega) = \sup_{x \in \Omega} p(x) \text{ and } p^- = p^-(\Omega) = \inf_{x \in \Omega} p(x).$$

For any $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and for any measurable function u on Ω , a modular $\rho_{p(\cdot)} = \rho_{p(\cdot), \Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \rightarrow \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0; \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define a Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

where ∇u is the gradient of u , that is, $\nabla u = (\partial_1 u, \dots, \partial_N u)$, $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

The following three propositions are well known (see Fan et al. [22], Fan and Zhao [23], Zhao et al. [39]).

Proposition 2.1. *Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$, and let $u, u_n \in L^{p(\cdot)}(\Omega)$ ($n = 1, 2, \dots$). Then we have the following properties.*

- (i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \iff \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.
- (ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \implies \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+}$.
- (iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \implies \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}$.
- (iv) $\lim_{n \rightarrow \infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \iff \lim_{n \rightarrow \infty} \rho_{p(\cdot)}(u_n - u) = 0$.
- (v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \rightarrow \infty \text{ as } n \rightarrow \infty \iff \rho_{p(\cdot)}(u_n) \rightarrow \infty \text{ as } n \rightarrow \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let $p \in C_+(\overline{\Omega})$, where*

$$C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}); p^- > 1\}.$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have

$$\int_{\Omega} |u(x)v(x)| dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \leq 2 \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, for any $p \in C_+(\overline{\Omega})$, $p'(\cdot)$ denote the conjugate exponent of $p(\cdot)$, that is, $p'(x) = p(x)/(p(x) - 1)$.

For $p \in C_+(\overline{\Omega})$, define for $x \in \overline{\Omega}$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.3. *Let Ω be a bounded domain of $\mathbb{R}^N$ with $C^{0,1}$ -boundary and let $p \in C_+(\overline{\Omega})$. Then we have the following properties.*

(i) *The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If $q(x) \in C(\overline{\Omega})$ with $q^- \geq 1$ satisfies that $q(x) \leq p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, where $\hookrightarrow$ means that the embedding is continuous.*

(iii) *If $q(x) \in C(\overline{\Omega})$ with $q^- \geq 1$ satisfies that $q(x) < p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.*

Next we consider the trace (cf. Fan [20]). Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$ -boundary Γ and $p \in C(\overline{\Omega})$ with $p^- \geq 1$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}(\Omega)$, the trace $\gamma(u) = u|_{\Gamma}$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1(\Gamma)$. We define

$$(\text{Tr}W^{1,p(\cdot)})(\Gamma) = \{f; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|f\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F|_{\Gamma} = f\}$$

for $f \in (\text{Tr}W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then we can see that $(\text{Tr}W^{1,p(\cdot)})(\Gamma)$ is a Banach space. In the later we also write $F|_{\Gamma} = g$ by $F = g$ on Γ . Moreover, for $i = 1, 2$, we denote

$$(\text{Tr}W^{1,p(\cdot)})(\Gamma_i) = \{f|_{\Gamma_i}; f \in (\text{Tr}W^{1,p(\cdot)})(\Gamma)\}$$

equipped with the norm

$$\|g\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\|f\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma)}; f \in (\text{Tr}W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f|_{\Gamma_i} = g\},$$

where the infimum can also be achieved, so for any $g \in (\text{Tr}W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F|_{\Gamma_i} = g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)} = \|g\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q \in C_+(\Gamma) := \{q \in C(\Gamma); q^- > 1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma_x$. We define

$$L^{q(\cdot)}(\Gamma) = \left\{ u; u : \Gamma \rightarrow \mathbb{R} \text{ is a measurable function with respect to } d\sigma_x \right. \\ \left. \text{satisfying } \int_{\Gamma} |u(x)|^{q(x)} d\sigma_x < \infty \right\}$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf \left\{ \lambda > 0; \int_{\Gamma} \left| \frac{u(x)}{\lambda} \right|^{q(x)} d\sigma_x \leq 1 \right\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_{\Gamma} |u(x)|^{q(x)} d\sigma_x.$$

Similarly as Proposition 2.1, we have the following proposition.

Proposition 2.4. *Let $q \in C(\Gamma)$ with $q^- \geq 1$, and let $u, u_n \in L^{q(\cdot)}(\Gamma)$. Then we have the following properties.*

- (i) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 (= 1, > 1) \iff \rho_{q(\cdot),\Gamma}(u) < 1 (= 1, > 1)$.
- (ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} > 1 \implies \|u\|_{L^{q(\cdot)}(\Gamma)}^{q^-} \leq \rho_{q(\cdot),\Gamma}(u) \leq \|u\|_{L^{q(\cdot)}(\Gamma)}^{q^+}$.
- (iii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \implies \|u\|_{L^{q(\cdot)}(\Gamma)}^{q^+} \leq \rho_{q(\cdot),\Gamma}(u) \leq \|u\|_{L^{q(\cdot)}(\Gamma)}^{q^-}$.
- (iv) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \rightarrow 0 \iff \rho_{q(\cdot),\Gamma}(u_n) \rightarrow 0$.
- (v) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \rightarrow \infty \iff \rho_{q(\cdot),\Gamma}(u_n) \rightarrow \infty$.

The Hölder inequality also holds for functions on Γ .

Proposition 2.5. *Let $q \in C(\Gamma)$ with $q^- > 1$. Then the following inequality holds.*

$$\int_{\Gamma} |f(x)g(x)| d\sigma_x \leq 2\|f\|_{L^{q(\cdot)}(\Gamma)}\|g\|_{L^{q'(\cdot)}(\Gamma)} \text{ for all } f \in L^{q(\cdot)}(\Gamma), g \in L^{q'(\cdot)}(\Gamma).$$

Proposition 2.6. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$ -boundary Γ and let $p \in C_+(\overline{\Omega})$. If $f \in (\text{Tr}W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(\Gamma)} \leq C\|f\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma)}.$$

In particular, If $f \in (\text{Tr}W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma_i)$ and $\|f\|_{L^{p(\cdot)}(\Gamma_i)} \leq C\|f\|_{(\text{Tr}W^{1,p(\cdot)})(\Gamma)}$ for $i = 1, 2$.

For $p \in C_+(\overline{\Omega})$, define for $x \in \overline{\Omega}$,

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

The following proposition follows from Yao [37, Proposition 2.6].

Proposition 2.7. *Let $p \in C_+(\overline{\Omega})$. Then if $q \in C_+(\Gamma)$ satisfies $q(x) \leq p^\partial(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \rightarrow L^{q(\cdot)}(\Gamma)$ is well-defined and continuous and*

$$\|u\|_{L^{q(\cdot)}(\Gamma)} \leq C \|u\|_{W^{1,p(\cdot)}(\Omega)} \text{ for } u \in W^{1,p(\cdot)}(\Omega)$$

for some constant $C > 0$.

In particular, if $q(x) < p^\partial(x)$ for all $x \in \Gamma_2$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \rightarrow L^{q(\cdot)}(\Gamma)$ is compact.

Now we consider the weighted variable exponent Lebesgue space. Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and let $a(x)$ be a measurable function on Ω with $a(x) > 0$ a.e. $x \in \Omega$. We define a modular

$$\rho_{(p(\cdot), a(\cdot))}(u) = \int_{\Omega} a(x) |u(x)|^{p(x)} dx \text{ for any measurable function } u \text{ in } \Omega.$$

Then the weighted Lebesgue space is defined by

$$L_{a(\cdot)}^{p(\cdot)}(\Omega) = \{u; u \text{ is a measurable function on } \Omega \text{ satisfying } \rho_{(p(\cdot), a(\cdot))}(u) < \infty\}$$

equipped with the norm

$$\|u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0; \int_{\Omega} a(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Then $L_{a(\cdot)}^{p(\cdot)}(\Omega)$ is a Banach space.

We have the following proposition (cf. Fan [19, Proposition 2.5]).

Proposition 2.8. *Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$. For $u, u_n \in L_{a(\cdot)}^{p(\cdot)}(\Omega)$, we have the following.*

- (i) For $u \neq 0$, $\|u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)} = \lambda \iff \rho_{(p(\cdot), a(\cdot))}\left(\frac{u}{\lambda}\right) = 1$.
- (ii) $\|u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \iff \rho_{(p(\cdot), a(\cdot))}(u) < 1 (= 1, > 1)$.
- (iii) $\|u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)} > 1 \implies \|u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)}^{p^-} \leq \rho_{(p(\cdot), a(\cdot))}(u) \leq \|u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)}^{p^+}$.
- (iv) $\|u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)} < 1 \implies \|u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)}^{p^+} \leq \rho_{(p(\cdot), a(\cdot))}(u) \leq \|u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)}^{p^-}$.
- (v) $\lim_{n \rightarrow \infty} \|u_n - u\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)} = 0 \iff \lim_{n \rightarrow \infty} \rho_{(p(\cdot), a(\cdot))}(u_n - u) = 0$.
- (vi) $\|u_n\|_{L_{a(\cdot)}^{p(\cdot)}(\Omega)} \rightarrow \infty \text{ as } n \rightarrow \infty \iff \rho_{(p(\cdot), a(\cdot))}(u_n) \rightarrow \infty \text{ as } n \rightarrow \infty$.

The author of [19] also derived the following proposition (cf. [19, Theorem 2.1]).

Proposition 2.9. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$ -boundary and $p \in C_+(\overline{\Omega})$. Moreover, let $a \in L^{\alpha(\cdot)}(\Omega)$ satisfy $a(x) > 0$ a.e. $x \in \Omega$ and $\alpha \in C_+(\overline{\Omega})$. If $q \in C(\overline{\Omega})$ satisfies*

$$1 \leq q(x) < \frac{\alpha(x) - 1}{\alpha(x)} p^*(x) \text{ for all } x \in \overline{\Omega},$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L_{a(\cdot)}^{q(\cdot)}(\Omega)$ is compact.

Similarly, let $q \in C(\Gamma)$ with $q^- \geq 1$ and let $b(x)$ be a measurable function with respect to σ on Γ with $b(x) > 0$ σ -a.e. $x \in \Gamma$. We define a modular

$$\rho_{(q(\cdot), b(\cdot)), \Gamma}(u) = \int_{\Gamma} b(x) |u(x)|^{q(x)} d\sigma_x.$$

Then the weighted Lebesgue space on Γ is defined by

$$L_{b(\cdot)}^{q(\cdot)}(\Gamma) = \{u; u \text{ is a } \sigma\text{-measurable function on } \Gamma \text{ satisfying } \rho_{(q(\cdot), b(\cdot)), \Gamma}(u) < \infty\}$$

equipped with the norm

$$\|u\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)} = \inf \left\{ \lambda > 0; \int_{\Gamma} b(x) \left| \frac{u(x)}{\lambda} \right|^{q(x)} d\sigma_x \leq 1 \right\}.$$

Then $L_{b(\cdot)}^{q(\cdot)}(\Gamma)$ is a Banach space.

Then we have the following proposition.

Proposition 2.10. *Let $q \in C(\Gamma)$ with $q^- \geq 1$. For $u, u_n \in L_{b(\cdot)}^{q(\cdot)}(\Gamma)$, we have the following.*

- (i) $\|u\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)} < 1 (= 1, > 1) \iff \rho_{(q(\cdot), b(\cdot)), \Gamma}(u) < 1 (= 1, > 1).$
- (ii) $\|u\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)} > 1 \implies \|u\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)}^{q^-} \leq \rho_{(q(\cdot), b(\cdot)), \Gamma}(u) \leq \|u\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)}^{q^+}.$
- (iii) $\|u\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)} < 1 \implies \|u\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)}^{q^+} \leq \rho_{(q(\cdot), b(\cdot)), \Gamma}(u) \leq \|u\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)}^{q^-}.$
- (iv) $\lim_{n \rightarrow \infty} \|u_n - u\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)} = 0 \iff \lim_{n \rightarrow \infty} \rho_{(q(\cdot), b(\cdot)), \Gamma}(u_n - u) = 0.$
- (v) $\|u_n\|_{L_{b(\cdot)}^{q(\cdot)}(\Gamma)} \rightarrow \infty \text{ as } n \rightarrow \infty \iff \rho_{(q(\cdot), b(\cdot)), \Gamma}(u_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$

The following proposition plays an important role in the present paper.

Proposition 2.11. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$ -boundary Γ and let $p \in C_+(\overline{\Omega})$. Assume that $0 < b \in L^{\beta(\cdot)}(\Gamma)$, $\beta \in C_+(\Gamma)$. If $r \in C(\Gamma)$ satisfies*

$$1 \leq r(x) < \frac{\beta(x) - 1}{\beta(x)} p^\partial(x) \text{ for all } x \in \Gamma,$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L_{b(\cdot)}^{r(\cdot)}(\Gamma)$ is compact.

For the proof, see Aramaki [6, Proposition 2.11].

Define a space by

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \quad (2.1)$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We get the following Poincaré-type inequality (cf. Ciarlet and Dinca [12]).

Proposition 2.12. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$ -boundary and let $p \in C_+(\overline{\Omega})$. Then there exists a constant $C = C(\Omega, N, p) > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \text{ for all } u \in X.$$

In particular, $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for $u \in X$.

For the direct proof, see Aramaki [1, Lemma 2.5].

Thus we can define the norm on X so that

$$\|v\|_X = \|\nabla v\|_{L^{p(\cdot)}(\Omega)} \text{ for } v \in X, \quad (2.2)$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Proposition 2.12.

3 Assumptions to the main theorems

In this section, we state the assumptions to the main theorems. Let $p \in C_+(\overline{\Omega})$ be fixed.

(A.0) Let $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a function satisfying that for a.e. $x \in \Omega$, the function $A(x, \cdot) : \mathbb{R}^N \ni \xi \mapsto A(x, \xi)$ is of C^1 -class, and for all $\xi \in \mathbb{R}^N$, the function $A(\cdot, \xi) : \Omega \ni x \mapsto A(x, \xi)$ is measurable. Moreover, suppose that $A(x, \mathbf{0}) = 0$ and put $\mathbf{a}(x, \xi) = \nabla_\xi A(x, \xi)$. Then $\mathbf{a}(x, \xi)$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$.

Assume that there exist constants $C_0, k_0 > 0$ and nonnegative functions $h_0 \in L^{p'(\cdot)}(\Omega)$ and $h_1 \in L^1(\Omega)$ with $h_1(x) \geq 1$ for a.e. $x \in \Omega$ such that the following conditions hold.

(A.1) $|\mathbf{a}(x, \xi)| \leq C_0(h_0(x) + h_1(x)|\xi|^{p(x)-1})$ for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A.2) $\mathbf{a}(x, \mathbf{0}) = \mathbf{0}$ for a.e. $x \in \Omega$ and

$$(\mathbf{a}(x, \xi) - \mathbf{a}(x, \eta)) \cdot (\xi - \eta) \geq \begin{cases} k_0 h_1(x) |\xi - \eta|^{p(x)} & \text{if } p(x) \geq 2, \\ k_0 h_1(x) (1 + |\xi| + |\eta|)^{p(x)-2} |\xi - \eta|^2 & \text{if } p(x) < 2 \end{cases}$$

for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$.

(A3) A is $p(\cdot)$ -subhomogeneous in the sense of

$$\mathbf{a}(x, \xi) \cdot \xi \leq p(x) A(x, \xi) + h_1(x) \text{ for all } \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega.$$

Example 3.1. Let $p \in C_+(\overline{\Omega})$.

- (i) $A(x, \xi) = \frac{h(x)}{p(x)} |\xi|^{p(x)}$ and $h \in L^1(\Omega)$ satisfying $h(x) \geq 1$ for a.e. $x \in \Omega$.
- (ii) $A(x, \xi) = \frac{h(x)}{p(x)} ((1 + |\xi|^2)^{p(x)/2} - 1)$ and $h \in L^{p'(\cdot)}(\Omega)$ satisfying $h(x) \geq 1$ for a.e. $x \in \Omega$.

Then $A(x, \xi)$ and $\mathbf{a}(x, \xi) = \nabla_{\xi} A(x, \xi)$ satisfy (A.0)-(A.3).

Remark 3.2. (i) When $h(x) \equiv 1$, (i) corresponds to the $p(\cdot)$ -Laplacian and (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

(ii) In many papers (for example, [38], [29], Aramaki [3, 6]), the authors assume that $\mathbf{a}(x, \xi) \cdot \xi \leq p(x)A(x, \xi)$ instead of (A.3). However, in the above Example 3.1 we see that if the example (ii) satisfies $1 < p(x) < 2$ in a subset of Ω with positive measure, then we have to assume (A.3).

Lemma 3.3. Under (A.0) and (A.2), we have the following.

- (i) We have

$$A(x, \xi) \geq \begin{cases} \frac{1}{p^+} k_0 h_1(x) |\xi|^{p(x)} & \text{if } p(x) \geq 2 \\ \frac{1}{2} k_0 h_1(x) (1 + |\xi|)^{p(x)-2} |\xi|^2 & \text{if } p(x) < 2. \end{cases}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

- (ii) There exists a constant $c > 0$ such that

$$\begin{aligned} & \frac{1}{2} A(x, \xi) + \frac{1}{2} A(x, \eta) - A\left(x, \frac{\xi + \eta}{2}\right) \\ & \geq \begin{cases} c h_1(x) |\xi - \eta|^{p(x)} & \text{if } p(x) \geq 2, \\ c h_1(x) (1 + |\xi| + |\eta|)^{p(x)-2} |\xi - \eta|^2 & \text{if } p(x) < 2 \end{cases} \end{aligned}$$

for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$.

In particular, $A(x, \xi)$ is convex with respect to ξ .

Proof. (i) It follows from (A.0) and (A.2) that

$$\begin{aligned} A(x, \xi) &= A(x, \xi) - A(x, \mathbf{0}) = \int_0^1 \frac{1}{dt} A(x, t\xi) dt = \int_0^1 \frac{1}{t} \mathbf{a}(x, t\xi) \cdot t\xi dt \\ &\geq \begin{cases} k_0 h_1(x) \int_0^1 t^{p(x)-1} |\xi|^{p(x)} dt & \text{if } p(x) \geq 2 \\ k_0 h_1(x) \int_0^1 (1 + t|\xi|)^{p(x)-2} t |\xi|^2 dt & \text{if } p(x) < 2. \end{cases} \\ &\geq \begin{cases} \frac{1}{p(x)} k_0 h_1(x) |\xi|^{p(x)} & \text{if } p(x) \geq 2 \\ \frac{1}{2} k_0 h_1(x) (1 + |\xi|)^{p(x)-2} |\xi|^2 & \text{if } p(x) < 2. \end{cases} \end{aligned}$$

For the proof of (ii), see Aramaki[7, Lemma 3.1]. □

Lemma 3.4. *Under (A.0)-(A.2), we have the following.*

- (i) $|A(x, \xi)| \leq C_0(h_0(x)|\xi| + h_1(x)|\xi|^{p(x)})$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.
- (ii) *There exist constants $c > 0$ and $C \geq 0$ such that*

$$\mathbf{a}(x, \xi) \cdot \xi \geq ch_1(x)|\xi|^{p(x)} - Ch_1(x) \text{ for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N.$$

In particular, if $p^- \geq 2$, then we can take $C = 0$.

For the proof, see [7, Lemma 3.4].

For the function $h_1 \in L^1(\Omega)$ with $h_1(x) \geq 1$ for a.e. $x \in \Omega$, we define a modular

$$\rho_{p(\cdot), h_1(\cdot)}(v) = \rho_{p(\cdot), h_1(\cdot), \Omega}(v) = \int_{\Omega} h_1(x) |\nabla v(x)|^{p(x)} dx \text{ for } v \in Y,$$

where Y is our basic space defined by

$$Y = \{v \in X; \rho_{p(\cdot), h_1(\cdot)}(v) < \infty\} \quad (3.1)$$

equipped with the norm

$$\|v\|_Y = \inf \left\{ \lambda > 0; \rho_{p(\cdot), h_1(\cdot)}\left(\frac{v}{\lambda}\right) \leq 1 \right\}.$$

Then Y is a Banach space (see Proposition 3.6 below). We note that $C_0^\infty(\Omega) \subset Y$. Since

$$\rho_{p(\cdot), h_1(\cdot)}(v) = \rho_{p(\cdot)}(h_1^{1/p(\cdot)} \nabla v),$$

we have

$$\|v\|_Y = \|h_1^{1/p(\cdot)} \nabla v\|_{L^{p(\cdot)}(\Omega)}. \quad (3.2)$$

Then we have the following lemma.

Lemma 3.5. (i) $Y \hookrightarrow X$ and $\|v\|_X \leq \|v\|_Y$ for all $v \in Y$.

(ii) Let $v \in Y$. Then $\|v\|_Y > 1 (= 1, < 1) \iff \rho_{p(\cdot), h_1(\cdot)}(v) > 1 (= 1, < 1)$.

(iii) Let $v \in Y$. Then $\|v\|_Y > 1 \implies \|v\|_Y^{p^-} \leq \rho_{p(\cdot), h_1(\cdot)}(v) \leq \|v\|_Y^{p^+}$.

(iv) Let $v \in Y$. Then $\|v\|_Y < 1 \implies \|v\|_Y^{p^+} \leq \rho_{p(\cdot), h_1(\cdot)}(v) \leq \|v\|_Y^{p^-}$.

(v) Let $u_n, u \in Y$. Then $\lim_{n \rightarrow \infty} \|u_n - u\|_Y = 0 \iff \lim_{n \rightarrow \infty} \rho_{p(\cdot), h_1(\cdot)}(u_n - u) = 0$.

(vi) Let $u_n \in Y$. Then $\|u_n\|_Y \rightarrow \infty$ as $n \rightarrow \infty \iff \rho_{p(\cdot), h_1(\cdot)}(u_n) \rightarrow \infty$ as $n \rightarrow \infty$.

Proposition 3.6. *The space $(Y, \|\cdot\|_Y)$ is a separable and reflexive Banach space.*

For the proof, see [3, Lemma 2.12].

The following proposition fulfills an important role in this paper. Put $\Omega_1 = \{x \in \Omega; p(x) \geq 2\}$, $\Omega_2 = \{x \in \Omega; p(x) < 2\}$.

Proposition 3.7. *Under (A.0)-(A.2), there exist positive constants c and C such that*

$$\begin{aligned} \int_{\Omega} (\mathbf{a}(x, \nabla u(x)) - \mathbf{a}(x, \nabla v(x))) \cdot (\nabla u(x) - \nabla v(x)) dx &\geq c \rho_{p(\cdot), h_1(\cdot), \Omega_1}(u - v) \\ &+ \left\{ c(C + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2} \rho_{p(\cdot), h_1(\cdot), \Omega_2}(u - v) \right\}^{2/p^+(\Omega_2)} \\ &\wedge \left\{ c(C + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2} \rho_{p(\cdot), h_1(\cdot), \Omega_2}(u - v) \right\}^{2/p^-(\Omega_2)} \end{aligned}$$

for $u, v \in Y$. Here and from now on, we denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for real numbers a and b .

In particular, if $v = 0$ and $\|u\|_Y < 1$, then we have

$$\begin{aligned} \int_{\Omega} \mathbf{a}(x, \nabla u(x)) \cdot \nabla u(x) dx &\geq c_1 (\rho_{p(\cdot), h_1(\cdot), \Omega_1}(u) + \rho_{p(\cdot), h_1(\cdot), \Omega_2}(u)^{2/p^-(\Omega_2)}) \\ &\geq c_2 \|u\|_Y^{2p^+/p^-} \end{aligned}$$

for some positive constants c_1 and c_2 .

We also get the following estimate.

$$\int_{\Omega} \mathbf{a}(x, \nabla u(x)) \cdot \nabla u(x) dx \geq c \|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} - C \|h_1\|_{L^1(\Omega)} \text{ for all } u \in Y \quad (3.3)$$

for some constants $c > 0$ and $C \geq 0$.

For the proof, see Aramaki [7, Proposition 3.7] (cf. [8, Proposition 3.5]).

Remark 3.8. (i) This proposition is firstly derived by the author [8], using a version of the idea of Glowinski and Marroco [24] who treated the case $p(x) = p = \text{const.}$.

(ii) Using Lemma 3.3 (i), we also get an estimate that there exists a constant $c > 0$ such that

$$\int_{\Omega} A(x, \nabla u(x)) dx \geq c \|u\|_Y^{2p^+/p^-} \text{ for } u \in Y \text{ with } \|u\|_Y < 1.$$

Throughout this paper, we consider the most general case where $|\Omega_1| > 0$ and $|\Omega_2| > 0$, where $|A|$ denotes the volume of any measurable set A .

Next we state the assumptions of the function $M = M(t)$ in (1.1).

(M.0) $M = M(t)$ is a continuous and monotone non-decreasing function on $[0, \infty)$.

(M.1) There exist constants $m_0, m_1 > 0$ and $k \geq l \geq 1$ such that

$$m_0 t^{l-1} \leq M(t) \leq m_1 (1 + t^{k-1}) \text{ for } t \geq 0.$$

(M.2) There exists $\mu \in (0, 1)$ such that $\widehat{M}(t) \geq (1 - \mu)M(t)t$ for $t \geq 0$, where

$$\widehat{M}(t) = \int_0^t M(s)ds. \quad (3.4)$$

We note that $\widehat{M} \in C^1((0, \infty)) \cap C([0, \infty))$, $\widehat{M}$ is convex, strictly monotone increasing and

$$\frac{m_0}{l}t^l \leq \widehat{M}(t) \leq m_1 \left(t + \frac{1}{k}t^k \right) \text{ for } t \geq 0. \quad (3.5)$$

We continue to state the assumptions of f and g in (1.1).

(f.1) $f = f(x, t)$ is a real Carathéodory function on $\Omega \times \mathbb{R}$ and there exist $1 \leq a \in L^{\alpha(\cdot)}(\Omega)$ with $\alpha \in C_+(\overline{\Omega})$ and $q \in C_+(\overline{\Omega})$ with

$$q(x) < \frac{\alpha(x) - 1}{\alpha(x)} p^*(x) \text{ for all } x \in \overline{\Omega}$$

such that $|f(x, t)| \leq C_1(1 + a(x)|t|^{q(x)-1})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$, where C_1 is a positive constant and $2lp^+/p^- < q^-$.

(f.2) There exist $\theta > (p^+/(1 - \mu)) \vee 2kp^+/p^-$ and $t_0 > 0$ such that

$$0 < \theta F(x, t) \leq f(x, t)t \text{ for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and a.e. } x \in \Omega,$$

where

$$F(x, t) = \int_0^t f(x, s)ds. \quad (3.6)$$

(f.3) $f(x, t) = o(|t|^{2lp^+/p^- - 1})$ uniformly in x as $t \rightarrow 0$.

(g.1) $g = g(x, t)$ is a real Carathéodory function on $\Gamma_2 \times \mathbb{R}$ and there exist $1 \leq b \in L^{\beta(\cdot)}(\Gamma_2)$ with $\beta \in C_+(\overline{\Gamma_2})$ and $r \in C_+(\overline{\Gamma_2})$ with

$$r(x) < \frac{\beta(x) - 1}{\beta(x)} p^\partial(x) \text{ for all } x \in \overline{\Gamma_2}$$

such that $|g(x, t)| \leq C_2(1 + b(x)|t|^{r(x)-1})$ for all $t \in \mathbb{R}$ and σ -a.e. $x \in \Gamma_2$, where C_2 is a positive constant and $2lp^+/p^- < r^-$.

(g.2) Let θ and t_0 be as in (f.2). That is, there exist $\theta > (p^+/(1 - \mu)) \vee 2kp^+/p^-$ and $t_0 > 0$ such that

$$0 < \theta G(x, t) \leq g(x, t)t \text{ for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and } \sigma\text{-a.e. } x \in \Gamma_2,$$

where

$$G(x, t) = \int_0^t g(x, s)ds. \quad (3.7)$$

(g.3) $g(x, t) = o(|t|^{2lp^+/p^- - 1})$ uniformly in x as $t \rightarrow 0$.

Lemma 3.9. *Under (f.1)-(f.3) and (g.1)-(g.3), we have the following.*

(i) *For any $\lambda > 0$, there exists a constant $C'_1 > 0$ such that*

$$|F(x, t)| \leq \lambda \frac{p^-}{2lp^+} |t|^{2lp^+/p^-} + C'_1 a(x) |t|^{q(x)} \text{ for a.e. } x \in \Omega, t \in \mathbb{R}.$$

(ii) *For any $\lambda > 0$, there exists a constant $C'_2 > 0$ such that*

$$|G(x, t)| \leq \lambda \frac{p^-}{2lp^+} |t|^{2lp^+/p^-} + C'_2 b(x) |t|^{r(x)} \text{ for } \sigma\text{-a.e. } x \in \Gamma_2, t \in \mathbb{R}.$$

Proof. From (f.3), for any $\lambda > 0$, there exists $\delta \in (0, 1)$ such that

$$|f(x, t)| \leq \lambda |t|^{2lp^+/p^- - 1} \text{ for a.e. } x \in \Omega, t \in (-\delta, \delta).$$

Hence we have

$$|F(x, t)| \leq \lambda \frac{p^-}{2lp^+} |t|^{2lp^+/p^-} \text{ for a.e. } x \in \Omega, t \in (-\delta, \delta).$$

On the other hand, from (f.1), we have

$$|F(x, t)| \leq C_1 \left(|t| + \frac{a(x)}{q(x)} |t|^{q(x)} \right) \leq C'_1 a(x) |t|^{q(x)} \text{ for a.e. } x \in \Omega, |t| \geq \delta.$$

Hence (i) follows. Similarly (ii) holds. $\square$

Define a functional on Y by

$$I(u) = \Psi(u) - J(u) - K(u), \quad \Psi(u) = \widehat{M}(\Phi(u)) \text{ for } u \in Y, \quad (3.8)$$

where

$$\Phi(u) = \int_{\Omega} A(x, \nabla u(x)) dx, \quad (3.9)$$

$$J(u) = \int_{\Omega} F(x, u(x)) dx, \quad F(x, t) \text{ is defined by (3.6),} \quad (3.10)$$

$$K(u) = \int_{\Gamma_2} G(x, u(x)) d\sigma_x, \quad G(x, t) \text{ is defined by (3.7).} \quad (3.11)$$

Proposition 3.10. *Assume that (M.0)-(M.2), (A.0)-(A.3), (f.1) and (g.1) hold. Then we have the following.*

(i) *The functionals J and K are sequentially weakly continuous in Y , that is, if $u_n \rightarrow u$ weakly in Y as $n \rightarrow \infty$, then $J(u_n) \rightarrow J(u)$ and $K(u_n) \rightarrow K(u)$ as $n \rightarrow \infty$.*

(ii) *The functional Ψ is sequentially weakly lower semi-continuous in Y , that is, if $u_n \rightarrow u$ weakly in Y as $n \rightarrow \infty$, then $\Psi(u) \leq \liminf_{n \rightarrow \infty} \Psi(u_n)$.*

Proof. By [3, Proposition 4.4], J and K are sequentially weakly continuous in Y and Φ is sequentially weakly lower semi-continuous in Y . We show that Ψ is sequentially weakly lower semi-continuous in Y . Let $u_n \rightarrow u$ weakly in Y as $n \rightarrow \infty$. Then

$$\Phi(u) \leq \liminf_{n \rightarrow \infty} \Phi(u_n) = \lim_{N \rightarrow \infty} \inf_{n \geq N} \Phi(u_n).$$

Since $\widehat{M}$ is monotone non-decreasing, we have $\widehat{M}(\inf_{n \geq N} \Phi(u_n)) \leq \widehat{M}(\Phi(u_n))$ for any $n \geq N$. Thus we have $\widehat{M}(\inf_{n \geq N} \Phi(u_n)) \leq \inf_{n \geq N} \widehat{M}(\Phi(u_n))$. Since $\widehat{M}$ is also continuous, we see that

$$\begin{aligned} \Psi(u) &= \widehat{M}(\Phi(u)) \leq \widehat{M}\left(\lim_{N \rightarrow \infty} \inf_{n \geq N} \Phi(u_n)\right) \\ &= \lim_{N \rightarrow \infty} \widehat{M}\left(\inf_{n \geq N} \Phi(u_n)\right) \leq \liminf_{n \rightarrow \infty} \widehat{M}(\Phi(u_n)) = \liminf_{n \rightarrow \infty} \Psi(u_n). \end{aligned}$$

This completes the proof. $\square$

Proposition 3.11. *Assume that (M.0)-(M.2), (A.0)-(A.3), (f.1) and (g.1) hold. Then we have the following.*

(i) *The functionals $\Psi, J, K \in C^1(Y, \mathbb{R})$ and the Fréchet derivatives Ψ', J' and K' satisfy the following equalities.*

$$\begin{aligned} \langle \Psi'(u), v \rangle &= M(\Phi(u)) \langle \Phi'(u), v \rangle = M(\Phi(u)) \int_{\Omega} \mathbf{a}(x, \nabla u(x)) \cdot \nabla v(x) dx, \\ \langle J'(u), v \rangle &= \int_{\Omega} f(x, u(x)) v(x) dx, \\ \langle K'(u), v \rangle &= \int_{\Gamma_2} g(x, u(x)) v(x) d\sigma_x \end{aligned}$$

for all $u, v \in Y$. Here and hereafter, we write the duality $\langle \cdot, \cdot \rangle_{Y^*, Y}$ by simply $\langle \cdot, \cdot \rangle$.

- (ii) *The functional Φ is coercive, that is, $\lim_{\|u\|_Y \rightarrow \infty} \Phi(u) = \infty$.*
- (iii) *The functional Φ is bounded on every bounded subset of Y .*
- (iv) *The operator $\Phi' : Y \rightarrow Y^*$ is coercive, that is,*

$$\lim_{\|u\|_Y \rightarrow \infty} \frac{\langle \Phi'(u), u \rangle}{\|u\|_Y} = \infty.$$

- (v) *Φ' is bounded on every bounded subset of Y .*

For the proof, see Aramaki [9, Proposition 3.8, 3.10 and 3.11].

Lemma 3.12. *Under (f.1)-(f.3) and (g.1)-(g.3), there exist constants c_1, c_2, C_3 and C_4 such that for $u \in Y$ with $\|u\|_Y < 1$, the following inequalities hold.*

(i) We have

$$J(u) \leq \lambda \frac{p^-}{2lp^+} c_1 \|u\|_Y^{2lp^+/p^-} + C_3 \|u\|_Y^{q^-},$$

where C is the constant of (3.3).

(ii) We have

$$K(u) \leq \lambda \frac{p^-}{2lp^+} c_2 \|u\|_Y^{2lp^+/p^-} + C_4 \|u\|_Y^{r^-}.$$

Proof. From Lemma 3.9 (i),

$$J(u) \leq \lambda \frac{p^-}{2lp^+} \int_{\Omega} |u(x)|^{2lp^+/p^-} dx + C_3 \int_{\Omega} a(x) |u(x)|^{q(x)} dx.$$

Here it suffices to note that since $2lp^+/p^- < q^- < p^*(x)$ for all $x \in \bar{\Omega}$, we have

$$\int_{\Omega} |u(x)|^{2lp^+/p^-} dx \leq C \|u\|_Y^{2lp^+/p^-}$$

with some constant $C > 0$, and

$$\int_{\Omega} a(x) |u(x)|^{q(x)} dx \leq C' \|u\|_Y^{q^-}.$$

(ii) follows from the similar arguments as (i). $\square$

Proposition 3.13. Assume that (M.0)-(M.2), (A.0)-(A.3), (f.1)-(f.3) and (g.1)-(g.3) hold. Then for any $\lambda > 0$, there exist constants $c, c_1, c_2 > 0$ and $C'_1, C'_2 > 0$ such that for $u \in Y$ with $\|u\|_Y < 1$ and any $\lambda > 0$,

$$I(u) \geq \left(\frac{cm_0}{l} - \lambda \frac{p^- c_1}{2lp^+} - \lambda \frac{p^- c_2}{2lp^+} \right) \|u\|_Y^{2lp^+/p^-} - C'_1 \|u\|_Y^{q^-} - C'_2 \|u\|_Y^{r^-}.$$

In particular, there exists $\rho \in (0, 1)$ such that

$$\inf_{\|u\|_Y = \rho} I(u) > 0. \quad (3.12)$$

Proof. Let $u \in Y$ with $\|u\|_Y < 1$. It follows from (3.4), (3.5), (A.2) and Proposition 3.7 with $v = 0$ and $\|u\|_Y < 1$ (cf. Remark 3.8 (ii)) that

$$\Psi(u) \geq \frac{m_0}{l} \left(\int_{\Omega} A(x, \nabla u(x)) dx \right)^l \geq \frac{cm_0}{l} \|u\|_Y^{2lp^+/p^-}.$$

From Lemma 3.12,

$$I(u) = \Psi(u) - J(u) - K(u) \geq \left(\frac{cm_0}{l} - \lambda \frac{p^- c_1}{2lp^+} - \lambda \frac{p^- c_2}{2lp^+} \right) \|u\|_Y^{2lp^+/p^-} - C_3 \|u\|_Y^{q^-} - C_4 \|u\|_Y^{r^-}.$$

If we choose $\lambda > 0$ small enough so that $c'' := \frac{cm_0}{l} - \lambda \frac{p^- c_1}{2lp^+} - \lambda \frac{p^- c_2}{2lp^+} > 0$, then we have

$$I(u) \geq \|u\|_Y^{2lp^+/p^-} (c'' - C_3 \|u\|_Y^{q^- - 2lp^+/p^-} - C_4 \|u\|_Y^{r^- - 2lp^+/p^-}).$$

Since $q^- > 2lp^+/p^-$ and $r^- > 2lp^+/p^-$, if $\|u\|_Y = \rho > 0$ is small, then we have $\inf_{\|u\|_Y = \rho} I(u) > 0$. $\square$

Proposition 3.14. *Assume that (M.0)-(M.2), (A.0)-(A.3), (f.1)-(f.3) and (g.1)-(g.3) hold. Then there exists a constant $C_5 > 0$ such that*

$$I(u) - \frac{1}{\theta} \langle I'(u), u \rangle \geq m_0 c \left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) \|u\|_Y^{p^-} - C_5 \text{ for all } u \in Y.$$

Proof. From (M.2), (A.3) and Lemma 3.4 (ii), for $u \in Y$, we have

$$\begin{aligned} \Psi(u) - \frac{1}{\theta} \langle \Psi'(u), u \rangle &= \widehat{M}(\Phi(u)) - \frac{1}{\theta} M(\Phi(u)) \langle \Phi'(u), u \rangle \\ &\geq (1-\mu) M(\Phi(u)) \Phi(u) - \frac{1}{\theta} M(\Phi(u)) \langle \Phi'(u), u \rangle \\ &\geq M(\Phi(u)) \left((1-\mu) \int_{\Omega} A(x, \nabla u(x)) dx - \frac{1}{\theta} \int_{\Omega} \mathbf{a}(x, \nabla u(x)) \cdot \nabla u(x) dx \right) \\ &\geq M(\Phi(u)) \left(\int_{\Omega} \left(\frac{1-\mu}{p(x)} - \frac{1}{\theta} \right) \mathbf{a}(x, \nabla u(x)) \cdot \nabla u(x) dx - (1-\mu) \|h_1/p\|_{L^1(\Omega)} \right) \\ &\geq M(\Phi(u)) \left(\left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) \langle \Phi'(u), u \rangle - (1-\mu) \|h_1/p\|_{L^1(\Omega)} \right). \end{aligned}$$

Since Φ' is coercive from Proposition 3.11 (iv) and $\theta > p^+/(1-\mu)$, there exists a large enough $M > 1$ such that if $\|u\|_Y \geq M$, then $\Phi(u) \geq 1$ and $((1-\mu)/p^+ - 1/\theta) \langle \Phi'(u), u \rangle - \|h_1/p\|_{L^1(\Omega)} \geq 0$ since Φ and Φ' are coercive. Hence using Lemma 3.4 (ii),

$$\begin{aligned} \Psi(u) - \frac{1}{\theta} \langle \Psi'(u), u \rangle &\geq m_0 \Phi(u)^{t-1} \left(\left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) \langle \Phi'(u), u \rangle - \|h_1/p\|_{L^1(\Omega)} \right) \\ &\geq m_0 \left(\left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) (c \|u\|_Y^{p^-} - C \|h_1/p\|_{L^1(\Omega)}) - \|h_1/p\|_{L^1(\Omega)} \right) \\ &\geq m_0 c \left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) \|u\|_Y^{p^-} - m_0 (C+1) \|h_1/p\|_{L^1(\Omega)}. \end{aligned}$$

When $\|u\|_Y \leq M$, since $\widetilde{M}$, M , Φ and Φ' is bounded on every bounded subset of Y from Proposition 3.11, Ψ and Ψ' are also bounded on every bounded subset

of Y . Thus we have

$$\left| \Psi(u) - \frac{1-\mu}{\theta} \langle \Psi'(u), u \rangle \right| \leq C'.$$

Hence for $\|u\|_Y \leq M$, we have

$$\Psi(u) - \frac{1}{\theta} \langle \Psi'(u), u \rangle \geq -C' \geq m_0 c \left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) \|u\|_Y^{p^-} - C''.$$

Therefore, we have

$$\Psi(u) - \frac{1}{\theta} \langle \Psi'(u), u \rangle \geq m_0 c \left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) \|u\|_Y^{p^-} - C_5''$$

for $u \in Y$.

On the other hand, it follows from (f.2) that

$$0 < \theta F(x, t) \leq f(x, t)t \text{ for a.e. } x \in \Omega, t \in \mathbb{R} \setminus (-t_0, t_0).$$

Put $\Omega_u = \{x \in \Omega; |u(x)| > t_0\}$. Then $\frac{1}{\theta} f(x, u(x))u(x) - F(x, u(x)) \geq 0$ for a.e. $x \in \Omega_u$. For $x \in \Omega \setminus \Omega_u$, we have

$$\left| \frac{1}{\theta} f(x, u(x))u(x) - F(x, u(x)) \right| \leq C_2(t_0 + a(x)t_0^{q^+} \vee t_0^{q^-}).$$

Hence we have

$$\begin{aligned} \frac{1}{\theta} \langle J'(u), u \rangle - J(u) &= \int_{\Omega_u} \left(\frac{1}{\theta} f(x, u(x))u(x) - F(x, u(x)) \right) dx \\ &\quad + \int_{\Omega \setminus \Omega_u} \left(\frac{1}{\theta} f(x, u(x))u(x) - F(x, u(x)) \right) dx \\ &\geq -C_2 \int_{\Omega \setminus \Omega_u} (t_0 + a(x)t_0^{q^+} \vee t_0^{q^-}) dx \\ &\geq -C_2 t_0 |\Omega| - C_2 t_0^{q^+} \vee t_0^{q^-} \|a\|_{L^1(\Omega)}. \end{aligned}$$

Similarly we have

$$\frac{1}{\theta} \langle K'(u), u \rangle - K(u) \geq -C_3 t_0 |\Gamma_2| - C_3 t_0^{r^+} \vee t_0^{r^-} \|b\|_{L^1(\Gamma_2)}.$$

Thus we have

$$\begin{aligned} I(u) - \frac{1}{\theta} \langle I'(u), u \rangle &= \Psi(u) - \frac{1}{\theta} \langle \Psi'(u), u \rangle - \left(J(u) - \frac{1}{\theta} \langle J'(u), u \rangle \right) \\ &\quad - \left(K(u) - \frac{1}{\theta} \langle K'(u), u \rangle \right) \geq \left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) m_0 c \|u\|_Y^{p^-} - C_5 \end{aligned}$$

for some constant C_5 . $\square$

Proposition 3.15. *Assume that (M.0)-(M.2), (A.0)-(A.3), (f.1)-(f.3) and (g.1)-(g.3) hold. Then the functional I satisfies the Palais-Smale condition, that is, if a sequence $\{u_n\} \subset Y$ satisfies that $\lim_{n \rightarrow \infty} I(u_n) = \gamma \in \mathbb{R}$ exists and $\lim_{n \rightarrow \infty} \|I'(u_n)\|_{Y^*} = 0$, then $\{u_n\}$ has a convergent subsequence.*

Proof. Let $\{u_n\} \subset Y$ satisfy that $\lim_{n \rightarrow \infty} I(u_n) = \gamma \in \mathbb{R}$ exists and $\lim_{n \rightarrow \infty} \|I'(u_n)\|_{Y^*} = 0$.

Step 1. The sequence $\{u_n\}$ is bounded in Y . Indeed, if it is false, then passing to a subsequence, we can assume that $\lim_{n \rightarrow \infty} \|u_n\|_Y = \infty$. By proposition 3.14, we have

$$I(u_n) \geq m_0 c \left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) \|u_n\|_Y^{p^-} - \frac{1}{\theta} \|I'(u_n)\|_{Y^*} \|u_n\|_Y - C_5$$

for large n . Since $\frac{1-\mu}{p^+} - \frac{1}{\theta} > 0$, $p^- > 1$ and $\lim_{n \rightarrow \infty} \|I'(u_n)\|_{Y^*} = 0$, we have $I(u_n) \rightarrow \infty$ as $n \rightarrow \infty$. This is a contradiction.

Step 2. Since Y is a reflexive Banach space from Proposition 3.6, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u \in Y$ such that $u_{n'} \rightarrow u$ weakly in Y as $n' \rightarrow \infty$. Since $\{u_{n'} - u\}$ is bounded in Y and $\lim_{n' \rightarrow \infty} \|I'(u_{n'})\|_{Y^*} = 0$, we see that

$$\langle I'(u_{n'}), u_{n'} - u \rangle \rightarrow 0 \text{ as } n' \rightarrow \infty.$$

By Propositions 2.9 and 2.11, $u_{n'} \rightarrow u$ strongly in $L_{a(\cdot)}^{q(\cdot)}(\Omega)$ and $L_{b(\cdot)}^{r(\cdot)}(\Gamma_2)$ as $n' \rightarrow \infty$. From (f.1), using the Hölder inequality,

$$\begin{aligned} & \left| \int_{\Omega} f(x, u_{n'}(x))(u_{n'}(x) - u(x)) dx \right| \\ & \leq \int_{\Omega} C_1 (1 + a(x)|u_{n'}(x)|^{q(x)-1}) |u_{n'}(x) - u(x)| dx \\ & \leq C_1 \int_{\Omega} (a(x)^{1/q(x)} |u_{n'}(x) - u(x)| \\ & \quad + a(x)^{1/q'(x)} |u_{n'}(x)|^{q(x)-1} a(x)^{1/q(x)} |u_{n'}(x) - u(x)|) dx \\ & \leq 2C_1 \|1\|_{L^{q'(\cdot)}(\Omega)} \|a^{1/q(\cdot)} |u_{n'} - u|\|_{L^{q(\cdot)}(\Omega)} \\ & \quad + 2C_1 \|a^{1/q'(\cdot)} |u_{n'}(\cdot)|^{q(\cdot)-1}\|_{L^{q'(\cdot)}(\Omega)} \|a^{1/q(\cdot)} |u_{n'} - u|\|_{L^{q(\cdot)}(\Omega)}. \end{aligned}$$

Since

$$\rho_{q'(\cdot)}(a^{1/q'(\cdot)} |u_{n'}|^{q(\cdot)-1}) = \int_{\Omega} a(x) |u_{n'}(x)|^{q(x)} dx$$

is bounded, we see that $\|a^{1/q'(\cdot)} |u_{n'}|^{q(\cdot)-1}\|_{L^{q'(\cdot)}(\Omega)}$ is bounded. Since $\|u_{n'} - u\|_{L_{a(\cdot)}^{q(\cdot)}(\Omega)} \rightarrow 0$ as $n' \rightarrow \infty$, we see that

$$\lim_{n' \rightarrow \infty} \langle J'(u_{n'}), u_{n'} - u \rangle = \lim_{n' \rightarrow \infty} \int_{\Omega} f(x, u_{n'}(x))(u_{n'}(x) - u(x)) dx = 0.$$

Similarly, we have

$$\lim_{n' \rightarrow \infty} \langle K'(u_{n'}), u_{n'} - u \rangle = \lim_{n' \rightarrow \infty} \int_{\Gamma_2} g(x, u_{n'}(x))(u_{n'}(x) - u(x)) d\sigma_x = 0.$$

Thus we have

$$\begin{aligned} & \lim_{n' \rightarrow \infty} \langle \Psi'(u_{n'}), u_{n'} - u \rangle \\ &= \lim_{n' \rightarrow \infty} (\langle J'(u_{n'}), u_{n'} - u \rangle + \langle K'(u_{n'}), u_{n'} - u \rangle + \langle I'(u_{n'}), u_{n'} - u \rangle) = 0. \end{aligned}$$

Since $|\langle \Psi'(u_{n'}), u_{n'} - u \rangle| \geq \frac{m_0}{l} |\langle \Phi'(u_{n'}), u_{n'} - u \rangle|^l$, we have $\lim_{n' \rightarrow \infty} \langle \Phi'(u_{n'}), u_{n'} - u \rangle = 0$. Since $u_{n'} \rightarrow u$ weakly in Y ,

$$\lim_{n' \rightarrow \infty} \langle \Phi'(u_{n'}) - \Phi'(u), u_{n'} - u \rangle = 0.$$

Since $\{u_{n'}\}$ is bounded in Y , it follows from Proposition 3.7 that

$$\int_{\Omega} h_1(x) |\nabla u_{n'}(x) - \nabla u(x)|^{p(x)} dx \rightarrow 0 \text{ as } n' \rightarrow \infty,$$

so $u_{n'} \rightarrow u$ strongly in Y . $\square$

4 Main theorems

In this section, we state the main theorems (Theorem 4.3, 4.5 and 4.6).

Definition 4.1. We say that $u \in Y$ is a weak solution of (1.1) if u satisfies that

$$\begin{aligned} & M \left(\int_{\Omega} A(y, \nabla u(y)) dy \right) \int_{\Omega} \mathbf{a}(x, \nabla u(x)) \cdot \nabla v(x) dx \\ &= \int_{\Omega} f(x, u(x)) v(x) dx + \int_{\Gamma_2} g(x, u(x)) v(x) d\sigma_x \text{ for all } v \in Y. \end{aligned} \quad (4.1)$$

Remark 4.2. Since $C_0^\infty(\Omega) \subset Y$, if $u \in Y$ satisfies (4.1), then the equation (1.1) holds in the distribution sense. Clearly we can see that $u \in Y$ is a weak solution of (1.1) if and only if u is a critical point of the functional I defined by (3.8), that is, $I'(u) = 0$.

Then we obtain the following three theorems.

Theorem 4.3. Let Ω be a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with a $C^{0,1}$ -boundary Γ satisfying (1.2). Under the hypotheses (M.0)-(M.2), (A.0)-(A.3), (f.1)-(f.3) and (g.1)-(g.3), the problem (1.1) has a nontrivial weak solution.

Remark 4.4. This theorem extends the result of Dai and Hao [14] in which the authors considered the case where $A(x, \xi) = \frac{1}{p(x)}|\xi|^{p(x)}$ and $\Gamma_2 = \emptyset$.

We impose one more assumption.

(f.4) Fix $\delta' \in (0, 1)$ to be sufficiently small. The function $f(x, t)$ satisfies the following inequality.

$$f(x, t) \geq \begin{cases} ct^{m-1} & \text{for } t \in [\delta', 1], \\ 0 & \text{for } t \in [0, \infty) \setminus [\delta', 1], \end{cases}$$

where $c > 0$ and $0 < m < 1$.

For example, A function $f(x, t) = \chi_{\delta'}(t)|t|^{m-2}t + a(x)|t|^{q(x)-2}t$, where $\chi_{\delta'} \in C_0(\mathbb{R})$ satisfying $0 \leq \chi_{\delta'} \leq 1$,

$$\chi_{\delta'}(t) = \begin{cases} 0 & \text{for } |t| \leq \delta'/2 \\ 1 & \text{for } \delta' \leq |t| \leq 1 \end{cases}$$

and a function $a = a(x)$ satisfies (f.1) verifies (f.1)-(f.4).

Theorem 4.5. *In addition to the hypotheses of Theorem 4.3, assume that (f.4) also holds. Then the problem (1.1) has at least two nontrivial weak solutions.*

Finally, in addition to the hypotheses of Theorem 4.3, we assume the following hypotheses.

(A.4) $A(x, \xi)$ is even with respect to ξ , that is, $A(x, -\xi) = A(x, \xi)$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

(f.5) $f(x, t)$ is odd, that is, $f(x, -t) = -f(x, t)$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

(g.4) $g(x, t)$ is odd, that is, $g(x, -t) = -g(x, t)$ for σ -a.e. $x \in \Gamma_2$ and all $t \in \mathbb{R}$.

Then we can derive that there exist infinitely many weak solution.

Theorem 4.6. *In addition to the hypotheses of Theorem 4.3, assume that (A.4), (f.5) and (g.4) also hold. Then the problem (1.1) has infinitely many nontrivial weak solutions.*

5 Proofs of Theorem 4.3, 4.5 and 4.6

In this section, we give proofs of Theorem 4.3, 4.5 and 4.6. Assume that (M.0)–(M.2), (A.1)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. In order to derive the theorems, we use the variational method.

Lemma 5.1. *Assume that (f.1)–(f.2) and (g.1)–(g.2) hold. Then we obtain the following.*

(i) $|F(x, t)| \leq C_1(|t| + \frac{a(x)}{q(x)}|t|^{q(x)}) \leq C'_1(1 + a(x)|t|^{q(x)})$ for a.e. $x \in \Omega$ and $t \in \mathbb{R}$.

- (ii) $|G(x, t)| \leq C_2(|t| + \frac{b(x)}{r(x)}|t|^{r(x)}) \leq C'_2(1 + b(x)|t|^{r(x)})$ for σ -a.e. $x \in \Gamma_2$ and $t \in \mathbb{R}$.
- (iii) There exists $\gamma \in L^{\alpha(\cdot)}(\Omega)$ such that $\gamma(x) > 0$ a.e. $x \in \Omega$ and $F(x, t) \geq \gamma(x)t^\theta$ for all $t \in [t_0, \infty)$ and a.e. $x \in \Omega$, where t_0 is the constant in (f.2).
- (iv) There exists $\delta \in L^{\beta(\cdot)}(\Gamma_2)$ such that $\delta(x) > 0$ σ -a.e. $x \in \Gamma_2$ and $G(x, t) \geq \delta(x)t^\theta$ for all $t \in [t_0, \infty)$ and σ -a.e. $x \in \Gamma_2$, where t_0 is the constant in (g.2).

Proof. (i) and (ii) easily follows from (f.1), (g.1) and the definitions (3.6), (3.7) of F , G , respectively.

(iii) From (f.2), for $t \geq t_0$,

$$0 < \theta F(x, t) \leq f(x, t)t \text{ for a.e. } x \in \Omega. \quad (5.1)$$

Put $\gamma(x) = F(x, t_0)t_0^{-\theta}$. Then $\gamma(x) > 0$ for a.e. $x \in \Omega$ and it follows from (i) that

$$\gamma(x) \leq C'_1(1 + a(x)t_0^{q(x)})t_0^{-\theta} \leq C'_1(1 + a(x)t_0^{q^+} \vee t_0^{q^-})t_0^{-\theta}.$$

So $\gamma \in L^{\alpha(\cdot)}(\Omega)$. From (5.1),

$$\frac{\theta}{\tau} \leq \frac{f(x, \tau)}{F(x, \tau)} = \frac{\frac{\partial F}{\partial \tau}(x, \tau)}{F(x, \tau)} \text{ for } \tau \geq t_0.$$

Integrating this inequality over (t_0, t) , we have

$$\theta \log \frac{t}{t_0} \leq \log \frac{F(x, t)}{F(x, t_0)} \text{ for all } t \geq t_0.$$

This implies that $F(x, t) \geq \gamma(x)t^\theta$ for all $t \geq t_0$.

(iv) follows from the similar argument as (iii) using (g.2). $\square$

5.1 Proof of Theorem 4.3

For a proof of Theorem 4.3, we apply the following standard Mountain-Pass Theorem (cf. Willem [35]).

Proposition 5.2. *Let $(V, \|\cdot\|_V)$ be a Banach space and $I \in C^1(V, \mathbb{R})$ be a functional satisfying the Palais-Smale condition. Assume that $I(0) = 0$, and there exist $\rho > 0$ and $z_0 \in V$ such that $\|z_0\|_V > \rho$, $I(z_0) \leq I(0) = 0$ and*

$$\alpha := \inf\{I(u); u \in V \text{ with } \|u\|_V = \rho\} > 0.$$

Put $G = \{\varphi \in C([0, 1], V); \varphi(0) = 0, \varphi(1) = z_0\}$ and $\beta = \inf\{\max I(\varphi([0, 1])); \varphi \in G\}$. Then $\beta \geq \alpha$ and β is a critical value of I .

We apply Proposition 5.2 with $(V, \|\cdot\|_V) = (Y, \|\cdot\|_Y)$. By Proposition 3.15, the functional I satisfies the Palais-Smale condition. By (3.12), there exists $\rho \in (0, 1)$ such that

$$\inf\{I(u); u \in Y \text{ with } \|u\|_Y = \rho\} > 0. \quad (5.2)$$

Since $\Psi(0) = J(0) = K(0) = 0$, we have $I(0) = 0$.

We show that there exists $u_0 \in Y$ such that $\|u_0\|_Y > \rho$ and $I(u_0) \leq 0$. Choose $v_0 \in C_0^\infty(\Omega)$ such that $v_0 \geq 0$ and $W = \{x \in \Omega; v_0(x) \geq t_0\}$ has a positive measure, where t_0 is as in (f.2). We see that $F(x, v_0(x)) > 0$ for a.e. $x \in W$ from (f.2). Let $t > 1$ and define $W_t = \{x \in \Omega; tv_0(x) \geq t_0\}$, then $W \subset W_t$. By Lemma 5.1 (iii), there exists $\gamma \in L^{\alpha(\cdot)}(\Omega) (\subset L^1(\Omega))$ such that $\gamma(x) > 0$ a.e. $x \in \Omega$ and $F(x, t) \geq \gamma(x)t^\theta$ for $t \in [t_0, \infty)$. Thereby,

$$\int_{W_t} F(x, tv_0(x))dx \geq \int_{W_t} \gamma(x)t^\theta v_0(x)^\theta dx \geq t^\theta L(v_0),$$

where $L(v_0) = \int_W \gamma(x)v_0(x)^\theta dx > 0$. For $t \in [0, t_0]$, $|F(x, t)| \leq C_1'(1 + a(x)t^{q(x)}) \leq C_1'(1 + a(x)t_0^{q^+} \vee t_0^{q^-})$. By (f.2), $F(x, st) \geq F(x, t)s^\theta$ for $t \in \mathbb{R} \setminus (-t_0, t_0)$ and $s > 1$. Indeed, if we define $h(s) = F(x, st)$, then

$$h'(s) = f(x, st)t = \frac{1}{s}f(x, st)st \geq \frac{\theta}{s}F(x, st) = \frac{\theta}{s}h(s).$$

Thus $h'(s)/h(s) \geq \theta/s$, so $\log(h(s)/h(1)) \geq \theta \log s$. This implies $h(s) \geq h(1)s^\theta$.

By (A.3) implies that

$$A(x, t\xi) + \frac{h_1(x)}{p(x)} \leq t^{p(x)} \left(A(x, \xi) + \frac{h_1(x)}{p(x)} \right) \text{ for } t > 1. \quad (5.3)$$

Indeed, put $k(s) = A(x, s\xi) + h_1(x)/p(x)$ for $s \geq 1$. Then we have

$$k'(s) = \frac{1}{s}a(x, s\xi) \cdot s\xi \leq \frac{p(x)}{s} \left(A(x, s\xi) + \frac{h_1(x)}{p(x)} \right) = \frac{p(x)}{s}k(s).$$

Thereby $k'(s)/k(s) \leq p(x)/s$ for $s \geq 1$. Integrating this inequality over $(1, t)$, we get (5.3). Thus we see that

$$\begin{aligned} \Phi(tu) &= \int_{\Omega} A(x, t\nabla u(x))dx \\ &\leq \int_{\Omega} \left\{ t^{p(x)} \left(A(x, \nabla u(x)) + \frac{h_1(x)}{p(x)} \right) - \frac{h_1(x)}{p(x)} \right\} dx \\ &\leq t^{p^+} \Phi(u) + (t^{p^+} - 1) \|h_1/p\|_{L^1(\Omega)} \\ &\leq t^{p^+} (\Phi(u) + C_6) \end{aligned}$$

for $u \in Y$ and $t > 1$, with some constant C_6 . Hence we have

$$\begin{aligned}
I(tv_0) &= \widehat{M}(\Phi(tv_0)) - J(tv_0) \\
&\leq m_1 \left(\Phi(tv_0) + \frac{1}{k} \Phi(tv_0)^k \right) - \int_{W_t} F(x, tv_0(x)) dx - \int_{\Omega \setminus W_t} F(x, tv_0(x)) dx \\
&\leq m_1 \left(t^{p^+} (\Phi(v_0) + C_6) + \frac{1}{k} t^{kp^+} (\Phi(v_0) + C_6)^k \right) \\
&\quad - t^\theta L(v_0) + C'_1 (|\Omega| + t_0^{q^+} \vee t_0^{q^-} \|a\|_{L^1(\Omega)}).
\end{aligned}$$

Since $\theta > kp^+ \geq p^+$ from (f.2) and $L(v_0) > 0$, we can see that $I(tv_0) \rightarrow -\infty$ as $t \rightarrow \infty$. Hence there exists $t_1 > 1$ such that $\|t_1 v_0\|_Y > \rho$ and $I(t_1 v_0) \leq 0$. Put $u_0 = t_1 v_0$.

If we define $\varphi(t) = tu_0$, then $\varphi \in G$, so $G \neq \emptyset$. Hence all the hypotheses of Proposition 5.2 hold. Therefore, $\beta = \inf\{\max I(\varphi([0, 1])); \varphi \in G\}$ satisfies that $\beta \geq \alpha > 0$ and β is a critical value of I , that is, there exists $u_1 \in Y$ such that $I(u_1) = \beta$ and $I'(u_1) = 0$. Thus u_1 is a weak solution of (1.1). Since $I(u_1) = \beta \geq \alpha > 0 = I(0)$, u_1 is a nontrivial weak solution of (1.1). This completes the proof of Theorem 4.3.

5.2 Proof of Theorem 4.5.

It follows from (f.4) that for $0 \leq t \leq 1$,

$$F(x, t) \geq \begin{cases} \int_{\delta'}^t f(x, s) ds & \text{if } t \geq \delta', \\ 0 & \text{if } t < \delta' \end{cases} \geq \begin{cases} \frac{c}{m} (t^m - (\delta')^m) & \text{if } t \geq \delta', \\ 0 & \text{if } t < \delta'. \end{cases}$$

Fix $t_1 \in (0, 1)$ so that small enough and choose $\delta' \in (0, 1)$ such that $(\delta')^m \leq t_1$. If $(\delta')^m \leq t$, then $F(x, t) \geq \frac{c}{m} (t^m - t)$ since $(\delta')^m \geq \delta'$. Choose $\varphi \in C_0^\infty(\Omega)$ so that $0 \leq \varphi \leq 1$ and $\varphi \not\equiv 0$. Put $\Omega_{\delta'} = \{x \in \Omega; (\delta')^m \leq t_1 \varphi(x)\}$. Then we note that $|\Omega \setminus \Omega_{\delta'}| \rightarrow 0$ as $\delta' \rightarrow 0$. Thus we have

$$\begin{aligned}
J(t_1 \varphi) &= \int_{\Omega} F(x, t_1 \varphi(x)) dx \\
&\geq \int_{\Omega_{\delta'}} F(x, t_1 \varphi(x)) dx \\
&\geq \frac{c}{m} \int_{\Omega_{\delta'}} ((t_1 \varphi(x))^m - t_1 \varphi(x)) dx \\
&\geq \frac{c}{m} t_1^m \left(\int_{\Omega} \varphi(x)^m dx - \int_{\Omega \setminus \Omega_{\delta'}} \varphi(x)^m dx \right) - \frac{c}{m} t_1 \int_{\Omega_{\delta'}} \varphi(x) dx \\
&\geq \frac{c}{m} t_1^m \left(\int_{\Omega} \varphi(x)^m dx - |\Omega \setminus \Omega_{\delta'}| \right) - \frac{c}{m} t_1 |\Omega|.
\end{aligned}$$

If we replace δ' with smaller one, if necessary, we may assume that $\int_{\Omega} \varphi(x)^m dx - |\Omega \setminus \Omega_{\delta'}| > 0$.

On the other hand, since $A(x, \xi)$ is convex with respect to ξ from Lemma 3.3 and $A(x, \mathbf{0}) = 0$, we have $A(x, t_1 \xi) = A(x, t_1 \xi + (1 - t_1) \mathbf{0}) \leq t_1 A(x, \xi)$. Thus

$$\Phi(t_1 \varphi) = \int_{\Omega} A(x, t_1 \nabla \varphi(x)) dx \leq t_1 \Phi(\varphi).$$

Since $\widehat{M}$ is convex and monotone non-decreasing, we have

$$\widehat{M}(\Phi(t_1 \varphi)) \leq \widehat{M}(t_1 \Phi(\varphi)) = \widehat{M}(t_1 \Phi(\varphi) + (1 - t_1) \Phi(0)) \leq t_1 \widehat{M}(\Phi(\varphi)).$$

Therefore, we have

$$\begin{aligned} I(t_1 \varphi) = \widehat{M}(\Phi(t_1 \varphi) - J(t_1 \varphi)) &\leq t_1 \left(\widehat{M}(\Phi(\varphi)) + \frac{c}{m} |\Omega| \right) \\ &\quad - \frac{c}{m} t_1^m \left(\int_{\Omega} \varphi(x)^m dx - |\Omega \setminus \Omega_{\delta'}| \right). \end{aligned}$$

Since $0 < m < 1$, if $t_1 > 0$ is small enough, then we see that $I(t_1 \varphi) < 0$. By Proposition 3.13, I is bounded from below on $\overline{B_{\rho}(0)}$, where $B_{\rho}(0) = \{v \in Y; \|v\|_Y < \rho\}$, ρ is as in (5.2). Hence

$$-\infty < \underline{c} := \inf_{v \in \overline{B_{\rho}(0)}} I(v) < 0.$$

Let $0 < \varepsilon < \inf_{v \in \partial B_{\rho}(0)} I(v) - \inf_{v \in \overline{B_{\rho}(0)}} I(v)$. Here we note that $\inf_{v \in \partial B_{\rho}(0)} I(v) > 0$ from (5.2). There exists $u \in \overline{B_{\rho}(0)}$ such that

$$\inf_{v \in \overline{B_{\rho}(0)}} I(v) \leq I(u) \leq \inf_{v \in \overline{B_{\rho}(0)}} I(v) + \varepsilon^2.$$

Since $\inf_{v \in \overline{B_{\rho}(0)}} I(v) < 0$, we can choose $u \in \overline{B_{\rho}(0)}$ so that $I(u) < 0$. By applying the Ekeland variational principle (Ekeland [18, Theorem 1.1]) to the complete metric space $\overline{B_{\rho}(0)}$, there exists $u_{\varepsilon} \in \overline{B_{\rho}(0)}$ such that

$$I(u_{\varepsilon}) \leq I(u), \tag{5.4}$$

$$I(u_{\varepsilon}) \leq I(v) + \varepsilon \|v - u_{\varepsilon}\|_Y \text{ for all } v \in \overline{B_{\rho}(0)}, \tag{5.5}$$

$$\|u - u_{\varepsilon}\|_Y \leq \varepsilon. \tag{5.6}$$

Define a functional $\widehat{I} : \overline{B_{\rho}(0)} \rightarrow \mathbb{R}$ by $\widehat{I}(v) = I(v) + \varepsilon \|v - u_{\varepsilon}\|_Y$ for $v \in \overline{B_{\rho}(0)}$. Since $I(u_{\varepsilon}) \leq I(u) < 0$ from (5.4) and $I(v) > 0$ for all $v \in \partial B_{\rho}(0)$, we have $u_{\varepsilon} \in B_{\rho}(0)$. Choose $\rho' > 0$ small enough so that $u_{\varepsilon} + w \in \overline{B_{\rho}(0)}$ for $w \in \overline{B_{\rho'}(0)}$.

From (5.5), since $\widehat{I}(u_\varepsilon) \leq \widehat{I}(u_\varepsilon + w)$ for all $w \in \overline{B_{\rho'}(0)}$, we have

$$\begin{aligned} & \frac{\langle I'(u_\varepsilon), w \rangle + \varepsilon \|w\|_Y}{\|w\|_Y} \\ &= \frac{\langle I'(u_\varepsilon), tw \rangle + \varepsilon t \|w\|_Y - (\widehat{I}(u_\varepsilon + tw) - \widehat{I}(u_\varepsilon))}{t \|w\|_Y} + \frac{\widehat{I}(u_\varepsilon + tw) - \widehat{I}(u_\varepsilon)}{t \|w\|_Y}. \end{aligned}$$

Here we note that from (5.5),

$$\widehat{I}(u_\varepsilon + tw) - \widehat{I}(u_\varepsilon) = I(u_\varepsilon + tw) + \varepsilon \|tw\|_Y - I(u_\varepsilon) \geq 0$$

for $t \in (0, 1)$. Hence

$$\frac{\langle I'(u_\varepsilon), w \rangle + \varepsilon \|w\|_Y}{\|w\|_Y} \geq \frac{\langle I'(u_\varepsilon), tw \rangle - (I(u_\varepsilon + tw) - I(u_\varepsilon))}{t \|w\|_Y} \rightarrow 0 \text{ as } t \rightarrow +0.$$

So $\langle I'(u_\varepsilon), w \rangle + \varepsilon \|w\|_Y \geq 0$ for all $w \in \overline{B_{\rho'}(0)}$, so $\langle I'(u_\varepsilon), w \rangle \geq -\varepsilon \|w\|_Y$. Replacing w with $-w$, we have $|\langle I'(u_\varepsilon), w \rangle| \leq \varepsilon \|w\|_Y$ for all $w \in \overline{B_{\rho'}(0)}$. Thus $\|I'(u_\varepsilon)\|_{Y^*} \leq \varepsilon$. Letting $\varepsilon \rightarrow 0$, we see that $I(u_\varepsilon) \rightarrow \underline{c}$ and $I'(u_\varepsilon) \rightarrow 0$ in Y^* . Since I satisfies the Palais-Smale condition in Y and $I \in C^1(Y, \mathbb{R})$, there exist a subsequence $\{u_n\}$ of $\{u_\varepsilon\}$ and $u_2 \in \overline{B_\rho(0)}$ such that $u_n \rightarrow u_2$ in Y and $I'(u_2) = 0$. Therefore, u_2 is a weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 = I(0)$, u_2 is a nontrivial weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 = I(0) < I(u_1)$, we have $u_1 \neq u_2$. This completes the proof of Theorem 4.5.

5.3 Proof of Theorem 4.6

We apply the following Symmetric Mountain-Pass Theorem due to the Rabinowitz [32, Theorem 9.12] (cf. Xie and Xiao [36, Proposition 2.1]).

Proposition 5.3. *Let V be an infinite-dimensional real Banach space. A functional $I : V \rightarrow \mathbb{R}$ is of C^1 -class and satisfies the Palais-Smale condition. Furthermore, assume that*

(I.1) $I(0) = 0$ and I is an even functional, that is, $I(-u) = I(u)$ for all $u \in V$.

(I.2) There exist positive constants α and ρ such that

$$\inf_{u \in \partial B_\rho(0)} I(u) \geq \alpha.$$

(I.3) For each finite-dimensional linear subspace $V_1 \subset V$, the set $\{u \in V_1; I(u) \geq 0\}$ is bounded.

Then I has an unbounded sequence of critical values.

We apply Proposition 5.3 with $V = Y$. Note that the functional I defined by (3.8) is of class C^1 (Proposition 3.11 (i)) and satisfies the Palais-Smale condition (Proposition 3.15). From (A.4), (f.5) and (g.4), (I.1) is trivial. (I.2) follows from (3.12). Thus it suffices to derive (I.3).

Let $u \in Y$ with $\|u\|_Y > 1$. Since it follows from (3.5) that

$$\Psi(u) = \widehat{M}(\Phi(u)) \leq m_1 \left(\Phi(u) + \frac{1}{k} \Phi(u)^k \right),$$

$\Phi(u) \leq c_1 \|h_0\|_{L^{p'(\cdot)}(\Omega)} \|u\|_Y + C_1 \|u\|_Y^{p^+}$ and $k \geq 1$, we have $\Psi(u) \leq C_5 \|u\|_Y^{kp^+}$ for some constant $C_5 > 0$. Since $F(x, t)$ is even with respect to t , it follows from Lemma 5.1 (iii) that $F(x, t) \geq \gamma(x)|t|^\theta$ for $|t| \geq t_0$. Define $\Omega_{t_0} = \{x \in \Omega; |u(x)| \geq t_0\}$. Then

$$J(u) = \int_{\Omega} F(x, u(x)) dx = \int_{\Omega_{t_0}} F(x, u(x)) dx + \int_{\Omega \setminus \Omega_{t_0}} F(x, u(x)) dx.$$

From Lemma 5.1 (i),

$$\int_{\Omega \setminus \Omega_{t_0}} |F(x, u(x))| dx \leq C'_1 |\Omega| + t_0^{q^+} \vee t_0^{q^-} \|a\|_{L^1(\Omega)} =: C'_2.$$

Hence we have

$$\begin{aligned} J(u) &\geq \int_{\Omega_{t_0}} \gamma(x) |u(x)|^\theta dx - C'_2 \\ &= \int_{\Omega} \gamma(x) |u(x)|^\theta dx - \int_{\Omega \setminus \Omega_{t_0}} \gamma(x) |u(x)|^\theta dx - C'_2 \\ &\geq \int_{\Omega} \gamma(x) |u(x)|^\theta dx - C_7, \end{aligned}$$

where C_7 is a constant. Similarly we have

$$K(u) \geq \int_{\Gamma_2} \delta(x) |u(x)|^\theta d\sigma_x - C_8,$$

where C_8 is a constant.

We note that

$$\left(\int_{\Omega} \gamma(x) |u(x)|^\theta dx + \int_{\Gamma_2} \delta(x) |u(x)|^\theta d\sigma \right)^{1/\theta} \quad (5.7)$$

is a norm in Y .

Let Y_1 be any finite-dimensional linear subspace of Y . Since Y_1 is of finite-dimensional, the above norm is equivalent to the norm $\|u\|_Y$ in Y_1 , so there exists $C_9 > 0$ such that

$$C_9 \|u\|_Y^\theta \leq \int_{\Omega} \gamma(x) |u(x)|^\theta dx + \int_{\Gamma_2} \delta(x) |u(x)|^\theta d\sigma_x.$$

Therefore, for $u \in Y_1$ with $\|u\|_Y > 1$, we have

$$I(u) \leq C_5 \|u\|_Y^{kp^+} - C_9 \|u\|_Y^\theta + C_7 + C_8.$$

If $u \in Y_1$ with $\|u\|_Y > 1$ satisfies $I(u) \geq 0$, then we have $C_9 \|u\|_Y^\theta \leq C_5 \|u\|_Y^{kp^+} + C_6 + C_7$. Since $\theta > kp^+$, the set $\{u \in Y_1; \|u\|_Y > 1, I(u) \geq 0\}$ is bounded, so $\{u \in Y_1; I(u) \geq 0\}$ is bounded.

Since all the assumptions of Proposition 5.3 hold, I has an unbounded sequence of critical values, so problem (1.1) has infinitely many weak solutions. This completes the proof of Theorem 4.6.

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