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BIIDEALS IN INVOLUTION RINGS AND SEMI-GROUPS

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Abstract

In a semiprime ring (semigroup with 0), every minimal quasiideal is a minimal biideal and vice versa. Moreover, for prime *-semigroups with 0, each *-minimal *-biideal is a minimal *-biideal. Nevertheless, A *-biideal B of a prime *-ring (*-semigroup) A is minimal if and only if B has the form $B = RR^*$, for a minimal right ideal R of A. Furthermore, for a semiprime *-semigroup S, each *-minimal *-biideal B is either minimal or a direct union of a minimal biideal C of S and its involutive image C^*. Finally, a set of equivalent conditions are given for a *-simple *-semigroup S to be the union of its *-minimal *-biideals.

1 Introduction

All rings considered are associative and a semigroup S will always mean a semigroup with an element 0 satisfying $0s = s0 = 0$, for all $s \in S$.

Recall that a nonzero idempotent f of a semigroup S is called *primitive* if for every nonzero idempotent $e \in S$, the relation $ef = fe = e$ implies $f = e$. An element $a \in S$ is *regular*, in the sense of von Neumann, if $a \in aAa$.

A *quasiideal* Q of a ring (semigroup) A is an additive subgroup (nonempty subset) of A satisfying $QA \cap AQ \subseteq Q$. Since $Q^2 \subseteq QA \cap AQ \subseteq Q$, it follows that the quasiideal Q is a subring (subsemigroup) of A. A *biideal* B of a ring

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(semigroup) A is a subring (subsemigroup) of A satisfying $BAB \subseteq B$. By the way, each quasiideal Q is a biideal, since $QAQ \subseteq QA \cap AQ \subseteq Q$, but the converse is not true (see [8]).

A semigroup S is the *direct union* of its ideals A_γ, $\gamma \in \Gamma$; written as $S = \overset{\bullet}{\underset{\gamma\in\Gamma}{\cup}} A_\gamma$, if $S = \underset{\gamma\in\Gamma}{\cup} A_\gamma$ and $A_\gamma \cap (\underset{\delta\neq\gamma}{\cup} A_\delta) = 0$.

A ring A is said to be a **-ring* if there is an *involution* * on A satisfying

$$a^{**} = a, (ab)^* = b^*a^*, (a+b)^* = a^* + b^*$$

for all $a, b \in A$. Similarly, a *-semigroup is a semigroup with an involution satisfying the first two identities.

A **-ideal* (**-biideal*) B of a *-ring (*-semigroup) A will indicate a self adjoint ideal (biideal) B; that is $B^*= B$. A nonzero *-ideal (*-biideal) B of a *-ring (*-semigroup) A is said to be **-minimal* if B does not properly contain any nonzero *-ideal (*-biideal) of A (see [3], [4] and [5]). It is clear that a minimal *-ideal (*-biideal) is **-minimal*.

A *-ring (*-semigroup) A without identity is said to be **-simple* if $A^2 \neq 0$ and the only *-ideals of A are 0 and A. However, each simple *-ring (*-semigroup) is *-simple while the converse is not true (see [1]). A *-simple semigroup containing a primitive idempotent is called completely *-simple.

2 Semiprime rings and semigroups

Our first result shows that in semiprime rings (not necessarily with involution), each minimal quasiideal is a minimal biideal and vice versa.

Proposition 1. *Each minimal quasiideal of a semiprime ring is a minimal biideal and vice versa.*

Proof Let A be a semiprime ring and Q be a minimal quasiideal of A, then Q is a biideal and by [8, Corollary 7.3a], $Q = eA \cap Af = eAf$, where e and f are idempotents in A such that eA and Af are minimal right and minimal left ideals of A, respectively. To show that Q is a minimal biideal, let S be a nonzero biideal of A contained in Q. Hence $0 \neq S \subseteq Q = eAf \subseteq Af$, and consequently $AS \subseteq Af$. But Af is a minimal left ideal, so either $AS = 0$ or $AS = Af$. The first case is impossible because it implies that S is a nonzero left ideal with $S^2 = 0$, which contradicts the primeness of A. Thus $AS = Af$ and similarly $SA = eA$. Therefore, $Q = eAf = SAf = SAS \subseteq S$ which forces $Q = S$. The Converse is evident from [9, Theorem 5] and [8, Theorem 6.7a]. $\square$

The following corresponding result for semigroups can be proved similarly using [8, Corollary 7.3b], [9, Theorem 4] and [8, Theorem 6.7b].

Proposition 2. *Each minimal quasiideal of a semiprime semigroup is a minimal biideal and vice versa.*

Using Proposition 1, we can replace minimal quasiideal by minimal biideal in Propositions 7.6a and 7.6b, Corollaries 7.5a and 7.5b and Theorem 10.6 in [8] to get new adaptations for them.

3 Rings and semigroups with involution

In [7], Mendes proved in Proposition 5 that " a *-minimal *-biideal of a prime *-ring A is a minimal *-biideal of A" , By a similar proof, the corresponding result for semigroups can also be obtained as follows.

Proposition 3. *Every *-minimal *-biideal B of a prime *-semigroup S is a minimal *-biideal.*

However, the results of Mendes and that of Proposition 3 are not true for (semiprime) involution rings (semigroups) according to [6, Proposition 4] and Proposition 7, respectively.

For prime *-rings, the following proposition gives a necessary and sufficient condition for a *-biideal to be minimal.

Proposition 4. *A *-biideal B of a prime *-ring (*-semigroup) A is minimal if and only if B has the form $B = RR^*$, where R is a minimal right ideal of A .*

Proof Because A is prime, $A^2 \neq 0$. If B is a minimal *-biideal of the prime *-ring A, then $0 \neq BAB$ and $0 \neq BA^2B$ are biideals of A and by the minimality of B we get $B = BAB = BA^2B = (BA)(AB) = RR^*$, where $R = BA$ is a right ideal of A and $R^* = (BA)^* = AB$. To show that $R = BA$ is a minimal right ideal of A, let C be a right ideal of A such that $C \subseteq BA$. Then $CB \subseteq BAB \subseteq B$ is a biideal of A and the minimality of B implies $CB = B \subseteq C$ Thus $BA \subseteq CA \subseteq C$ and $C = BA$ follows. Conversely, let R be a minimal right ideal of A, then $B = RR^*$ is *-biideal . Since R^* is a minimal left ideal of A and $0 \neq RAR^* \subseteq RR^*$, it follows by [8, Theorem 6.7a] and Proposition 1 that B is a minimal *-biideal of A. □

Proposition 5. *Let A be a prime *-ring (*-semigroup) and B be a minimal *-biideal of A. Then for every $a \in A$, the subring aBa^* is either zero or a minimal *-biideal of A.*

Proof If $aBa^* \neq 0$, then $0 \neq aB$ is a minimal biideal of A, by [8, Proposition 7.6a] and Proposition 1. Apply [8, Proposition 7.6a] and Proposition 1 again, aBa^* is a minimal *-biideal, since it is closed under involution. □

By a similar proof to Corollary 3 in [6], if $\langle M\rangle$ denotes the ideal of a semigroup S generated by the subset M, we get the corresponding result for semiprime *-semigroups.

Lemma 6. *Let S be a semiprime *-semigroup. If M is a subset of S, then the following conditions are equivalent:*
1) $\langle M\rangle \cap \langle M^*\rangle = 0$,
2) $MSM^* = 0$,
3) $M^*SM = 0$,
4) $SM \cap M^*S = 0$,
5) $MS \cap SM^* = 0$.

The characterization of *-minimal *-biideals in semiprime *-semigroups is given as follows.

Proposition 7. *Let S be a semiprime *-semigroup. If B is a *-minimal *-biideal of S, then either B is a minimal *-biideal or $B = C \overset{\bullet}{\cup} C^*$, as a direct union, for a minimal biideal C of S.*

Proof S is semiprime and B is *-minimal imply $B^2 = B$ and $B = BSB$. If B is not minimal, then there exists a nonzero biideal $C \subsetneq B$. We claim that $CSC^* = 0$, otherwise $B = CSC^*$, by the *-minimality of B. Lemma 6 gives $C^*SC \neq 0$ and similarly $B = C^*SC$. Hence $B = BSB = CSC^*SC^*SC \subseteq CSC \subseteq C$ which contradicts the choice of C. Thus $CSC^* = C^*SC = 0$ and consequently $C \cup C^*$ is a *-biideal of S because

$$(C \cup C^*)S(C \cup C^*) \subseteq CSC \cup C^*SC^* \subseteq C \cup C^*.$$

Lemma 6 again gives $C \cap C^* = 0$. Since $C \overset{\bullet}{\cup} C^*$ is a nonzero *-biideal contained in B, the *-minimality of B implies $B = C \overset{\bullet}{\cup} C^*$. Finally, to show that C is a minimal biideal, let D be a biideal such that $D \subsetneq C$, whence $D \overset{\bullet}{\cup} D^* \subsetneq C \overset{\bullet}{\cup} C^* = B$. The *-minimality of B forces $D \overset{\bullet}{\cup} D^* = 0$ which implies $D = 0$. □

Proposition 8. *Let S be a semiprime *-semigroup. If C is a minimal biideal of S, then either C is a *-minimal *-biideal or $B = C \overset{\bullet}{\cup} C^*$ is a *-minimal *-biideal of S.*

Proof Clearly C is a *-minimal biideal. If C is not closed under involution, then $C \cap C^* = 0$ and $B = C \overset{\bullet}{\cup} C^*$ is a *-minimal *-biideal of S. □

To prove the main theorem of this section, we need the following auxiliary results.

Lemma 9. *Let e be a primitive idempotent of a *-semigroup S. If each element of the *-biideal e^*Se (eSe^*) is regular, then it is a *-minimal *-biideal of S.*

Proof Let B be a nonzero *-biideal contained in e^*Se and $0 \neq a \in B \subseteq e^*Se = e^*S \cap Se$. Since a is regular, it follows from Proposition 10.4 in [8] that $Se = Sa$ and $e^*S = aS$, whence $e^*Se = e^*Sa = aSa \subseteq BSB \subseteq B$. Thus $e^*Se = B$ is a *-minimal *-biideal. □

Corollary 10. *Let e be a primitive idempotent of the simple *- semigroup S. Then the *-biideal e^*Se (eSe^*) is *-minimal.*

Proof S is completely simple and by [8, Corollary 10.8], it is regular. Hence e^*Se (eSe^*) is regular and the result follows from Lemma 9. □

Finally, we prove some equivalent statements for which a *-simple semigroup is the union of its *-minimal *-biideals.

Theorem 11. *Let S be a *-simple *-semigroup. The following are equivalent conditions on S:*
1) S contains a primitive idempotent.
*2) S contains at least one *-minimal *-biideal.*
3) S contains a minimal right ideal possessing a nonzero idempotent element.
4) S contains a minimal left ideal possessing a nonzero idempotent element.
5) S is the union of its minimal left ideals and the union of its minimal right ideals.
6) S is the union of its minimal biideals.
*7) S is the union of its *-minimal *-biideals.*

Proof S is *-simple and by [2, Lemma 1], S is either simple or $S = K \dot{\cup} K^*$ as a direct union, where K is an ideal of S and is a simple subsemigroup. Moreover S is semiprime.
Consider first the case when S is simple.
1) $\Longrightarrow$ 2). If e is a primitive idempotent of S, then by Corollary 10, e^*Se is a *-minimal *-biideal of S.
2) $\Longrightarrow$ 3). Since S is semiprime, it has a minimal biideal, by Proposition 7, whence it has a minimal right ideal possessing a nonzero idempotent element, by [8, Corollary 7.5b] and Proposition 1.
3) $\Longrightarrow$ 4) is evident since the involutive image of a minimal right ideal is a minimal left ideal.
4) $\Longrightarrow$ 5) follows from [8, Theorem 10.6] and Proposition 1.
5) $\Longrightarrow$ 6) is direct by [8, Theorem 10.6] and Proposition 1.
6) $\Longrightarrow$ 7). If $S = \underset{\gamma\in\Gamma}{\cup} B_\gamma$ is the union of its minimal biideals B_γ, $\gamma \in \Gamma$, then $S = S^* = \underset{\gamma\in\Gamma}{\cup} B_\gamma^*$ and by Propositions 8, S is the union of *-minimal *-biideals. From Proposition 7, each *-minimal *-biideal of S is either a minimal *-biideal B_γ or $B_\delta \cup B_\delta^*$ for a minimal biideal B_δ.
7) $\Longrightarrow$ 1). Similar to the proof of the implication 6 $\Longrightarrow$ 7, S is the union of its

minimal biideals. Hence S possesses a primitive idempotent, by [8, Theorem 10.6] and Proposition 1.

Now, let $S = K \dot{\cup} K^*$, as a direct union, where K is an ideal of S and is a simple subsemigroup.
1) $\Longrightarrow$ 2). The primitive idempotent e is contained in either K or K^*. If $e \in K$, then by [8, Theorem 10.6] and Proposition 1, K has a minimal biideal C which is also a minimal biideal of S, by [8, Theorem 10.2]. Since C is not closed under involution it follows from Proposition 8 that $B = C \dot{\cup} C^*$ is a*-biideal of S. Moreover, B is *-minimal, because any *-biideal D of S contained in B has the form $D = I \dot{\cup} I^*$, where I is a nonzero biideal of S contained in C, which is impossible from the minimality of C.
2) $\Longrightarrow$ 3). By Proposition 7, S has a minimal biideal and [8, Theorem 10.6] and Proposition 1 give the result.
3) $\Longrightarrow$ 4) is evident from [8, Theorem 10.6].
4) $\Longrightarrow$ 5). Let L be a minimal left ideal of S possessing a nonzero idempotent element, then L is contained in either K or K^*. If $L \subseteq K$, then $L^* \subseteq K^*$ is a minimal right ideal of S possessing a nonzero idempotent element. Applying [8, Theorem 10.6], each of K and K^* is the union of its minimal left ideals and the union of its minimal right ideals. But each minimal left (right) ideal of S is contained in either K or K^* , so that S is the union of its minimal left ideals and the union of its minimal right ideals.
5) $\Longrightarrow$ 6). If $S = \bigcup_{\gamma\in\Gamma} L_\gamma$ is the union of its minimal left ideals L_γ, $\gamma \in \Gamma$, then $S = S^* = \bigcup_{\gamma\in\Gamma} L^*_\gamma$ is the union of its minimal right ideals. From $S = S^2 = (\bigcup_{\gamma\in\Gamma} L^*_\gamma)(\bigcup_{\gamma\in\Gamma} L_\gamma) \subseteq \bigcup_{\gamma\in\Gamma}\bigcup_{\delta\in\Gamma}(L^*_\gamma L_\delta)$, we get $S = \bigcup_{\gamma\in\Gamma}\bigcup_{\delta\in\Gamma}(L^*_\gamma L_\delta)$. Moreover, the product $L^*_\gamma L_\delta$, by [9, Theorem 4], is a minimal biideal of S and according to [9, Theorem 5], each minimal biideal of S has this form, so S is the union of its minimal biideals.
6) $\Longrightarrow$ 7). If $S = \bigcup_{\gamma\in\Gamma} B_\gamma$ is the union of its minimal biideals B_γ, $\gamma \in \Gamma$, then $S = S^* = \bigcup_{\gamma\in\Gamma} B^*_\gamma$ and by Propositions 8, S is the union of *-minimal *-biideals. From Proposition 7, each *-minimal *-biideal of S is either a minimal *-biideal B_γ or $B_\delta \cup B^*_\delta$ for a minimal biideal B_δ.
7) $\Longrightarrow$ 1). S has a *-minimal *-biideal $B = C \dot{\cup} C^*$, as a direct union, for a minimal biideal C of S, by Proposition 7 and the fact that $S = K \dot{\cup} K^*$. Further, $C \subseteq K$ or $C \subseteq K^*$, whence K or K^* contains a primitive idempotent, by [8, Theorem 10.6] and Proposition 1, which is a primitive idempotent of S. $\square$

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