

SOME PROPERTIES ON THE LIE DERIVATIVE OF LINEAR CONNECTIONS ON $\mathbb{R}^n$

Bui Cao Van* and Tran Thi Kim Ha†

**Department of Mathematics, Vinh University
182 Le Duan, Vinh City, Vietnam
email: buicaovan@gmail.com*

*†Central National Preparatory School
46 Nguyen Thien Thuat, Nha Trang City, Vietnam.
email: tranthikimhant@gmail.com*

Abstract

The aim of this work is to study some properties of the Lie derivative of the linear connection ∇ , the conjugate derivative d_{∇} with the linear connection and using them for searching the curvature, the torsion of a space $\mathbb{R}^n$ along the linear flat connection ∇ .

1 Introduction

The concept of Lie derivative appeared in the early 30s and was related to the works of Slebodzinski, Dantzig, Schouten and Van Campen ([8]). The problem consisted of generalization of the operations which has an invariant sense only when it is applied to a scalar field, to the case of tensor field and the connection object. The Lie differentiation theory plays an important role in studying automorphisms of differential geometric structures. In a more developed form, this theory is presented by K. Yano ([20]).

In some recent decades, the Lie derivative of forms and its application was investigated by many authors (see [7], [6], [5], [17],[18], [19],[20] and the references given therein). In 2010, Sultanov used the Lie derivative of the linear connection to study the curvature tensor and the torsion tensor on linear algebras (see [19], pp. 362-412). In 2012, basing on the Lie derivative of real-valued

Key words: Lie derivative, conjugate derivative, vector-valued differential forms, linear connection.

(2010) Mathematics Subject Classification: Primary 49Q20; Secondary 53A04, 53A40.

forms on the Riemannian n -dimensional manifold, N. H. Quang, K. P. Chi and B. C. Van constructed the Lie derivative of the currents on Riemann manifolds and given some applications on Lie groups (see [12]). The primary goal of our work is the extension of the operations of Lie derivative to objects defined on the vector-valued differential forms of a manifold. The main goal of the present work is to investigate some properties on the Lie derivative and the conjugate derivative d_{∇} on $\mathbb{R}^n$. For an application, we give some results for searching the curvature, the torsion of a space $\mathbb{R}^n$ along the linear flat connection ∇ .

2 Notation and Preliminaries

We denote the vector space of all smooth vector fields on $\mathbb{R}^n$ by $\mathfrak{B}(\mathbb{R}^n)$ and $\mathfrak{F}(\mathbb{R}^n)$ is the vector space of smooth functions on $\mathbb{R}^n$. Let ∇ be a linear connection on $\mathbb{R}^n$ and D be an usual directional derivative that give rise to a linear connection on $\mathbb{R}^n$. More precisely, if $X = X^i \partial_i$ and $Y = Y^j \partial_j$, then we define

$$\nabla_{X^i \partial_i} (Y^j \partial_j) = X^i \partial_i (Y^j) \partial_j$$

The torsion tensor R of $\mathbb{R}^n$ is defied by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \forall X, Y \in \mathfrak{B}(\mathbb{R}^n).$$

If $T = 0$, then the linear connection ∇ of $\mathbb{R}^n$ is said to be *flat*. Then D is the linear flat connection on $\mathbb{R}^n$.

The curvature tensor R of $\mathbb{R}^n$ is defied by

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

for any $X, Y, Z \in \mathfrak{B}(\mathbb{R}^n)$.

In particular, if $\nabla = D$, then we obtain $T = 0$ and $R = 0$. A vector field X is called parallel if $\nabla_Y X = 0, \forall Y \in \mathfrak{B}(\mathbb{R}^n)$.

Now, let F by any normed vector space and F has finite dimension m . A smooth differential k -form on $\mathbb{R}^n$ with values in F , for short, k -form on $\mathbb{R}^n$, is any smooth function, $\omega : \mathbb{R}^n \rightarrow \bigwedge^k(\mathbb{R}^n, F)$. The vector space of all k -forms on $\mathbb{R}^n$ is denoted $\Omega^k(\mathbb{R}^n, F)$. The vector space, $\Omega^*(\mathbb{R}^n, F) = \bigoplus_{k \geq 0} \Omega^k(\mathbb{R}^n, F)$, is

the set of differential forms on $\mathbb{R}^n$ with values in F (see [3], pp. 307).

Observe that $\Omega^0(\mathbb{R}^n, F) = C^\infty(\mathbb{R}^n, F)$, the vector space of smooth functions on $\mathbb{R}^n$ with values in F and $\Omega^1(\mathbb{R}^n, F) = C^\infty(\mathbb{R}^n, \text{Hom}(\mathbb{R}^n, F))$, the set of smooth functions from $\mathbb{R}^n$ to the set of linear maps from $\mathbb{R}^n$ to F . Also, $\Omega^k(\mathbb{R}^n, F) = (0)$ for $k > n$. Pick any basis, $(f_1, \dots, f_m)$, of F . Then, as every differential k -form, $\omega \in \Omega^k(\mathbb{R}^n, F)$, can be written uniquely as (see [3], pp.

311)

$$\omega = \sum_{i=1}^m \omega_i \cdot f_i, \quad (2.1)$$

where $\omega_1, \dots, \omega_m$ are smooth real-valued differential forms in $\Omega^k(\mathbb{R}^n, \mathbb{R})$ and we view f_i as the constant map with value f_i from $\mathbb{R}^n$ to F . Then, as

$$d\omega_p(X_p) = \sum_{i=1}^m (\omega'_i)_p(X_p) \cdot f_i, \quad (2.2)$$

for all $X_p \in T_p\mathbb{R}^n$, we see that (see [3], pp. 311)

$$d\omega = \sum_{i=1}^m d\omega_i \cdot f_i. \quad (2.3)$$

Actually, because $d\omega$ is defined independently of bases, the f_i do not need to be linearly independent; any choices of vectors and forms such that

$$\omega = \sum_{i=1}^p \omega_i \cdot f_i, \quad (2.4)$$

will do.

Let G, H be normal vector space. Given a bilinear map, $\phi : F \times G \rightarrow H$, a simple calculation shows that for all $\omega \in \Omega^k(\mathbb{R}^n, F)$ and for all $\eta \in \Omega^l(\mathbb{R}^n, G)$, we have (see [3], pp. 311)

$$\omega \wedge_\phi \eta = \sum_{i=1}^m \sum_{j=1}^{m'} \omega_i \wedge \eta_j \cdot \phi(f_i, g_j), \quad (2.5)$$

with $\omega = \sum_{i=1}^m \omega_i \cdot f_i$ and $\eta = \sum_{j=1}^{m'} \omega_j \cdot f_j$, where $(f_1, \dots, f_m)$ is a basis of F and $(g_1, \dots, g_{m'})$ is a basis of G .

If F, G, H are finite dimensions and $\phi : F \times G \rightarrow H$ is a bilinear map, then for all $\omega \in \Omega^k(\mathbb{R}^n, F)$ and for all $\eta \in \Omega^l(\mathbb{R}^n, G)$, we have (see [3], pp. 312)

$$d(\omega \wedge_\phi \eta) = d\omega \wedge_\phi \eta + (-1)^k \omega \wedge_\phi d\eta. \quad (2.6)$$

Similar to Equation (2.3), we have the inner product of the smooth vector-valued differential forms

$$X \lrcorner \omega = \sum_{i=1}^m (X \lrcorner \omega_i) \cdot f_i, \quad (2.7)$$

where $\omega = \sum_{i=1}^m \omega_i \cdot f_i$, and $X \lrcorner \omega_1, \dots, X \lrcorner \omega_m$ are the inner products of the smooth real-valued differential forms in $\Omega^k(\mathbb{R}^n, \mathbb{R})$ and $X \lrcorner \omega$ is given by:

$$(X \lrcorner \omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}), \quad (2.8)$$

for all $X_1, X_2, \dots, X_{k-1} \in \mathfrak{B}(\mathbb{R}^n)$.

If F, G, H are finite dimensions and $\phi : F \times G \rightarrow H$ is a bilinear map, then for all $\omega \in \Omega^k(\mathbb{R}^n, F)$ and all $\psi \in \Omega^l(\mathbb{R}^n, G)$, we easily get the following properties of the inner product of a smooth vector-valued differential form

$$X \lrcorner (\omega \wedge_\phi \psi) = (X \lrcorner \omega) \wedge_\phi \psi + (-1)^k \omega \wedge_\phi (X \lrcorner \psi). \quad (2.9)$$

We have known that the Cartans formula (see [6], pp.35) for the Lie derivative of real-valued differential forms on manifold states $L_X \omega = d(X \lrcorner \omega) + X \lrcorner (d\omega)$, $\forall \omega \in \Omega^k(\mathbb{R}^n)$, that is, $L_X = d \circ X \lrcorner + X \lrcorner \circ d$. The following Formula (2.10) gives the Cartans formula for the Lie derivative of vector-valued differential forms on $\mathbb{R}^n$. For all $\omega \in \Omega^k(\mathbb{R}^n, F)$, $\omega = \sum_{i=1}^m \omega_i \cdot f_i$, and $X \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned} d(X \lrcorner \omega) + X \lrcorner d\omega &= \sum_{i=1}^m d(X \lrcorner \omega_i) \cdot f_i + \sum_{i=1}^m (X \lrcorner d\omega_i) \cdot f_i \\ &= \sum_{i=1}^m [d(X \lrcorner \omega_i) + (X \lrcorner d\omega_i)] \cdot f_i \\ &= \sum_{i=1}^m (L_X \omega_i) \cdot f_i \\ &= L_X \omega \end{aligned} \quad (2.10)$$

For all $\omega \in \Omega^k(\mathbb{R}^n, F)$, and $X \in \mathfrak{B}(\mathbb{R}^n)$, we have the formula for the Lie derivative of the vector-valued differential k -forms ω on $\mathbb{R}^n$

$$(L_X \omega)(X_1, \dots, X_k) = L_X(\omega(X_1, \dots, X_k)) - \sum_{i=1}^k \omega(X_1, \dots, L_X X_i, \dots, X_k), \quad (2.11)$$

that is the analogous result in ([19], pp. 378).

3 The Lie derivative and the conjugate derivative on $\mathbb{R}^n$

In this section, we consider in case of the normed vector space $F = \mathfrak{B}(\mathbb{R}^n)$. Then a smooth differential k -form ω on $\mathbb{R}^n$ with values in $\mathfrak{B}(\mathbb{R}^n)$ is k -form linear, antisymmetric

$$\omega : \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{B}(\mathbb{R}^n) \times \dots \times \mathfrak{B}(\mathbb{R}^n) \rightarrow \mathfrak{B}(\mathbb{R}^n).$$

The main goal of the present work is to investigate some properties on the Lie derivative and the conjugate derivative d_{∇} on $\mathbb{R}^n$.

Definition 3.1. Given $X \in \mathfrak{B}(\mathbb{R}^n)$ and ∇ is a linear connection on $\mathbb{R}^n$. The map

$$L_X \nabla : \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{B}(\mathbb{R}^n) \rightarrow \mathfrak{B}(\mathbb{R}^n)$$

satisfying the condition

$$(L_X \nabla)(Y, Z) = L_X(\nabla_Y Z) - \nabla_{L_X Y} Z - \nabla_Y(L_X Z), \quad (3.1)$$

for all $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$ is called the Lie derivative of the linear connection ∇ along a vector field X .

Remark 3.2. For all $X_1, X_2 \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$L_{X_1+X_2} \nabla = L_{X_1} \nabla + L_{X_2} \nabla \quad (3.2)$$

Proof. For all $X_1, X_2 \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned} (L_{X_1+X_2} \nabla)(Y, Z) &= L_{X_1+X_2}(\nabla_Y Z) - \nabla_{L_{X_1+X_2} Y} Z - \nabla_Y(L_{X_1+X_2} Z) \\ &= [X_1 + X_2, \nabla_Y Z] - \nabla_{[X_1+X_2, Y]} Z - \nabla_Y([X_1 + X_2, Z]) \\ &= [X_1, \nabla_Y Z] + [X_2, \nabla_Y Z] - \nabla_{[X_1, Y]} Z - \nabla_{[X_2, Y]} Z - \\ &\quad - \nabla_Y([X_1, Z]) - \nabla_Y([X_2, Z]) \\ &= ([X_1, \nabla_Y Z] - \nabla_{[X_1, Y]} Z - \nabla_Y([X_1, Z])) + \\ &\quad + ([X_2, \nabla_Y Z] - \nabla_{[X_2, Y]} Z - \nabla_Y([X_2, Z])) \\ &= (L_{X_1} \nabla)(Y, Z) + (L_{X_2} \nabla)(Y, Z) \\ &= (L_{X_1} \nabla + L_{X_2} \nabla)(Y, Z). \end{aligned}$$

□

Proposition 3.3. Suppose that ∇ be a linear flat connection on $\mathbb{R}^n$. Then the map $L_X \nabla : \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{B}(\mathbb{R}^n) \rightarrow \mathfrak{B}(\mathbb{R}^n)$ is a bilinear.

Proof. For all $Y_1, Y_2, Z \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned} (L_X \nabla)(Y_1 + Y_2, Z) &= L_X(\nabla_{Y_1+Y_2} Z) - \nabla_{L_X Y_1+Y_2} Z - \nabla_{Y_1+Y_2}(L_X Z) \\ &= [X, \nabla_{Y_1+Y_2} Z] - \nabla_{[X, Y_1+Y_2]} Z - \nabla_{Y_1+Y_2}([X, Z]) \\ &= [X, \nabla_{Y_1} Z] + [X, \nabla_{Y_2} Z] - \nabla_{[X, Y_1]} Z - \nabla_{[X, Y_2]} Z \\ &\quad - \nabla_{Y_1}([X, Z]) - \nabla_{Y_2}([X, Z]) \\ &= (L_X \nabla)(Y_1, Z) + (L_X \nabla)(Y_2, Z). \end{aligned}$$

For all $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, $\varphi \in \mathfrak{F}(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, we have

$$\begin{aligned}
(L_X \nabla)(\varphi Y, Z) &= L_X(\nabla_{\varphi Y} Z) - \nabla_{L_X \varphi Y} Z - \nabla_{\varphi Y}(L_X Z) \\
&= [X, \nabla_{\varphi Y} Z] - \nabla_{[X, \varphi Y]} Z - \nabla_{\varphi Y}([X, Z]) \\
&= [X, \varphi \cdot \nabla_Y Z] - \nabla_{X[\varphi] \cdot Y + \varphi[X, Y]} Z - \varphi \cdot \nabla_Y([X, Z]) \\
&= X[\varphi] \cdot \nabla_Y Z + \varphi \cdot [X, \nabla_Y Z] - X[\varphi] \cdot \nabla_Y Z - \\
&\quad - \varphi \cdot \nabla_{[X, Y]} Z - \varphi \cdot \nabla_Y([X, Z]) \\
&= \varphi \cdot ([X, \nabla_Y Z] - \nabla_{[X, Y]} Z - \nabla_Y([X, Z])) \\
&= \varphi \cdot (L_X \nabla)(Y, Z).
\end{aligned}$$

For all $Y, Z_1, Z_2 \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}
(L_X \nabla)(Y, Z_1 + Z_2) &= L_X(\nabla_Y (Z_1 + Z_2)) - \nabla_{L_X Y} (Z_1 + Z_2) - \nabla_Y(L_X (Z_1 + Z_2)) \\
&= [X, \nabla_Y (Z_1 + Z_2)] - \nabla_{[X, Y]} (Z_1 + Z_2) - \nabla_Y([X, Z_1 + Z_2]) \\
&= [X, \nabla_Y Z_1] + [X, \nabla_Y Z_2] - \nabla_{[X, Y]} Z_1 - \nabla_{[X, Y]} Z_2 \\
&\quad - \nabla_Y([X, Z_1]) - \nabla_Y([X, Z_2]) \\
&= (L_X \nabla)(Y, Z_1) + (L_X \nabla)(Y, Z_2).
\end{aligned}$$

For all $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, $\varphi \in \mathfrak{F}(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, we have

$$\begin{aligned}
(L_X \nabla)(Y, \varphi \cdot Z) &= L_X(\nabla_Y (\varphi \cdot Z)) - \nabla_{L_X Y} (\varphi \cdot Z) - \nabla_Y(L_X (\varphi \cdot Z)) \\
&= [X, \nabla_Y (\varphi \cdot Z)] - \nabla_{[X, Y]} (\varphi \cdot Z) - \nabla_Y([X, \varphi \cdot Z]) \\
&= [X, Y[\varphi] \cdot Z + \varphi \cdot \nabla_Y Z] - [X, Y[\varphi] \cdot Z - \varphi \cdot \nabla_{[X, Y]} Z - \\
&\quad - \nabla_Y(X[\varphi] \cdot Z + \varphi \cdot [X, Z]) \\
&= [X, Y[\varphi] \cdot Z] + X[\varphi] \cdot \nabla_Y Z + \varphi \cdot [X, \nabla_Y Z] - [X, Y[\varphi] \cdot Z - \\
&\quad - \varphi \cdot \nabla_{[X, Y]} Z - \nabla_Y(X[\varphi] \cdot Z) - Y[\varphi] \cdot [X, Z] - \varphi \cdot \nabla_Y [X, Z] \\
&= \varphi \cdot (L_X \nabla)(Y, Z) + [X, Y[\varphi] \cdot Z] + X[\varphi] \cdot \nabla_Y Z - [X, Y[\varphi] \cdot Z - \\
&\quad - \nabla_Y(X[\varphi] \cdot Z) - Y[\varphi] \cdot [X, Z] \\
&= \varphi \cdot (L_X \nabla)(Y, Z) + X[Y[\varphi] \cdot Z] + Y[\varphi] \cdot [X, Z] + X[\varphi] \cdot \nabla_Y Z - \\
&\quad - [X, Y[\varphi] \cdot Z] - Y[X[\varphi] \cdot Z] - X[\varphi] \cdot \nabla_Y Z - Y[\varphi] \cdot [X, Z] \\
&= \varphi \cdot (L_X \nabla)(Y, Z).
\end{aligned}$$

□

Now, we note the linear flat connection ∇ and X is a parallel vector field on $\mathbb{R}^n$. Following theorem gives an application of the Lie derivative to determine the curvature of $\mathbb{R}^n$.

Theorem 3.4. *Suppose that ∇ be a linear flat connection and X be a parallel vector field on $\mathbb{R}^n$. Then we have*

$$(L_X \nabla)(Y, Z) = R(X, Y, Z), \forall Y, Z \in \mathfrak{B}(\mathbb{R}^n). \quad (3.3)$$

Proof. For any $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}
 (L_X \nabla)(Y, Z) &= L_X(\nabla_Y Z) - \nabla_{L_X Y} Z - \nabla_Y(L_X Z) \\
 &= [X, \nabla_Y Z] - \nabla_{L_X Y} Z - \nabla_Y([X, Z]) \\
 &= \nabla_X \nabla_Y Z - \nabla_{\nabla_Y X} Z - \nabla_{[X, Y]} Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X \\
 &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \\
 &= R(X, Y, Z).
 \end{aligned}$$

This proves the theorem. $\square$

Corollary 3.5. *Suppose that X be a parallel vector field on $\mathbb{R}^n$ and $\nabla = D$. Then we have $L_X D = 0$.*

Proof. By using Equation (3.3), we have

$$(L_X D)(Y, Z) = R(X, Y, Z) = 0, \forall Y, Z \in \mathfrak{B}(\mathbb{R}^n).$$

Hence, $L_X D = 0$. $\square$

Now, let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism and f_* be the push-forward of f . The mapping

$$f_* \nabla : \mathfrak{B}(\mathbb{R}^n) \times \mathfrak{B}(\mathbb{R}^n) \rightarrow \mathfrak{B}(\mathbb{R}^n)$$

is defined by the following formula:

$$(f_* \nabla)(f_* X, f_* Y) = f_*(\nabla_X Y), \forall X, Y \in \mathfrak{B}(\mathbb{R}^n). \quad (3.4)$$

Then $f_* \nabla$ is a linear connection on $\mathbb{R}^n$.

Proposition 3.6. *Suppose that ∇ be a linear connection on $\mathbb{R}^n$, $\tilde{X} = f_* X, \tilde{Y} = f_* Y, \forall X, Y \in \mathfrak{B}(\mathbb{R}^n)$. Then we have*

$$L_{f_* X} (f_* \nabla) (\tilde{X}, \tilde{Y}) = f_* ((L_X \nabla) (X, Y)) \quad (3.5)$$

Proof. For all $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}
 L_{f_* X} (f_* \nabla) (\tilde{X}, \tilde{Y}) &= (L_{f_* X} (f_* \nabla))(f_* Y, f_* Z) \\
 &= L_{f_* X} ((f_* \nabla)_{f_* Y} (f_* Z)) - (f_* \nabla)_{L_{f_* X} f_* Y} (f_* Z) - (f_* \nabla)_{f_* Y} (L_{f_* X} (f_* Z)) \\
 &= [f_* X, f_* (\nabla_Y Z)] - (f_* \nabla)_{f_* [X, Y]} (f_* Z) - (f_* \nabla)_{f_* Y} (f_* [X, Z]) \\
 &= f_* [X, \nabla_Y Z] - f_* (\nabla_{[X, Y]} Z) - f_* (\nabla_Y [X, Z]) \\
 &= f_* ([X, \nabla_Y Z] - \nabla_{[X, Y]} Z - \nabla_Y [X, Z]) \\
 &= f_* ((L_X \nabla)(Y, Z)).
 \end{aligned}$$

$\square$

Definition 3.7. Given a linear connection ∇ on M . The conjugate derivative with the connection ∇ on $\mathbb{R}^n$ is defined by the following formula:

$$\begin{aligned} (d_{\nabla}\omega)(X_0, X_1, \dots, X_k) &= \sum_{i=0}^k (-1)^i \nabla_{X_i}(\omega(X_0, X_1, \dots, \widehat{X_i}, \dots, X_k)) + \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k), \end{aligned} \quad (3.6)$$

for any $X_0, X_1, \dots, X_k \in \mathfrak{B}(\mathbb{R}^n)$ and the covariant derivative ∇_X of $\omega \in \Omega^k(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$ along a vector field X on $\mathbb{R}^n$ is defined by the formula:

$$(\nabla_X \omega)(X_1, \dots, X_k) = \nabla_X(\omega(X_1, \dots, X_k)) - \sum_{i=1}^k \omega(X_1, \dots, \nabla_X X_i, \dots, X_k), \quad (3.7)$$

for any $X_1, \dots, X_k \in \mathfrak{B}(\mathbb{R}^n)$

From definition 3.7, we have the following Remark.

Remark 3.8. Suppose that ∇ be a linear flat connection, ω be the constant differential 1-form and X be a vector field on $\mathbb{R}^n$. The identity mapping $I : \mathfrak{B}(\mathbb{R}^n) \rightarrow \mathfrak{B}(\mathbb{R}^n)$ is defined by $I(X) = X$, for all $X \in \mathfrak{B}(\mathbb{R}^n)$. The following identities holds:

- i) $L_X \omega = 0$;
- ii) $\nabla_X \omega = 0$;
- iii) $d_{\nabla} \omega = 0$;
- iv) $d_{\nabla} I = 0$.

Proof. i) For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$(L_X \omega)(Y) = [X, \omega(Y)] - \omega[X, Y] = \nabla_X(\omega(Y)) - \nabla_{\omega(Y)} X - \omega[X, Y].$$

On the other hand, since ω is a constant, we have $\omega = (\lambda_1 \omega_1, \dots, \lambda_n \omega_n)$, with $\lambda_i \in \mathbb{R}$ and $\omega_j = \sum_{i=1}^n \alpha_{ij} dx_i$, $\alpha_{ij} \in \mathbb{R}$. Hence,

$$\nabla_X(\omega(Y)) - \nabla_{\omega(Y)} X - \omega[X, Y] = 0.$$

Consequently $L_X \omega = 0$.

ii) For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$(\nabla_X \omega)(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y) = 0$$

iii) For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$d_{\nabla} \omega(X, Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X, Y]) = 0$$

iv) For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned} d_{\nabla} I(X, Y) &= \nabla_X(I(Y)) - \nabla_Y(I(X)) - I([X, Y]) \\ &= \nabla_X Y - \nabla_Y X - [X, Y] \\ &= T(X, Y) = 0 \end{aligned}$$

□

Proposition 3.9. *Suppose that ∇ be a linear flat connection on $\mathbb{R}^n$ and $\omega \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$. Then we have*

$$(d_{\nabla} \omega)(X, Y) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X), \forall X, Y \in \mathfrak{B}(\mathbb{R}^n). \quad (3.8)$$

Proof. For all $X, Y \in \mathfrak{B}(\mathbb{R}^n)$, we have

$$\begin{aligned} (\nabla_X \omega)(Y) &= \nabla_X(\omega(Y)) - \omega(\nabla_X Y); \\ (\nabla_Y \omega)(X) &= \nabla_Y(\omega(X)) - \omega(\nabla_Y X). \end{aligned}$$

Hence,

$$\begin{aligned} (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) &= \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega(\nabla_X Y - \nabla_Y X) \\ &= \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X, Y]) \\ &= (d_{\nabla} \omega)(X, Y), \forall X, Y \in \mathfrak{B}(\mathbb{R}^n). \end{aligned}$$

□

The following theorem gives a description the formula between the Lie derivative and the conjugate derivative d_{∇} on $\mathbb{R}^n$.

Theorem 3.10. *Let $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$ and $X \in \mathfrak{B}(\mathbb{R}^n)$. Then we have*

$$(L_X(d_{\nabla} \theta))(Y, Z) = (d_{\nabla}(L_X \theta))(Y, Z) + (L_X \nabla)(Y, \theta(Z)) - (L_X \nabla)(Z, \theta(Y)), \quad (3.9)$$

for all $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$.

Proof. For any $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$ and $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, we have

$$\begin{aligned} (L_X(d_{\nabla} \theta))(Y, Z) &= [X, (d_{\nabla} \theta)(Y, Z)] - (d_{\nabla} \theta)([X, Y], Z) - (d_{\nabla} \theta)(Y, [X, Z]) \\ &= [X, \nabla_Y(\theta(Z))] - [X, \nabla_Z(\theta(Y))] - [X, \theta(Y, Z)] - \\ &\quad - (\nabla_{[X, Y]}(\theta(Z)) - \nabla_Z(\theta([X, Y])) - \theta([X, Y], Z)) - \\ &\quad - (\nabla_Y(\theta([X, Z])) - \nabla_{[X, Z]}(\theta(Y)) - \theta([Y, [X, Z]])) - \\ &= [X, \nabla_Y(\theta(Z))] - [X, \nabla_Z(\theta(Y))] - [X, \theta(Y, Z)] - \\ &\quad - \nabla_{[X, Y]}(\theta(Z)) + \nabla_Z(\theta([X, Y])) + \theta([X, Y], Z) - \\ &\quad - \nabla_Y(\theta([X, Z])) + \nabla_{[X, Z]}(\theta(Y)) + \theta([Y, [X, Z]]). \end{aligned} \quad (3.10)$$

On the other hand, we have

$$\begin{aligned}
(d_{\nabla}(L_X\theta))(Y, Z) &= \nabla_Y((L_X\theta)(Z)) - \nabla_Z((L_X\theta)(Y)) - (L_X\theta)([Y, Z]) \\
&= \nabla_Y([X, \theta(Z)] - \theta([X, Z])) - \nabla_Z([X, \theta(Y)] - \theta([X, Y])) - \\
&\quad - ([X, \theta([Y, Z])] - \theta([X, ([Y, Z])])) \\
&= \nabla_Y([X, \theta(Z)]) - \nabla_Y(\theta([X, Z])) - \nabla_Z([X, \theta(Y)]) + \\
&\quad + \nabla_Z(\theta([X, Y])) - [X, \theta([Y, Z])] + \theta([X, ([Y, Z])]).
\end{aligned} \tag{3.11}$$

From Equations (3.10, 3.11), we obtain

$$\begin{aligned}
\left(L_X(d_{\nabla}\theta) - d_{\nabla}(L_X\theta) \right)(Y, Z) &= [X, \nabla_Y(\theta(Z))] - [X, \nabla_Z(\theta(Y))] \\
&\quad - [X, \theta(Y, Z)] - \nabla_{[X, Y]}(\theta(Z)) + \nabla_Z(\theta([X, Y])) + \theta([X, Y], Z) - \\
&\quad - \nabla_Y(\theta([X, Z])) + \nabla_{[X, Z]}(\theta(Y)) + \theta([Y, [X, Z]]) - \\
&\quad - \nabla_Y([X, \theta(Z)]) + \nabla_Y(\theta([X, Z])) + \nabla_Z([X, \theta(Y)]) - \\
&\quad - \nabla_Z(\theta([X, Y])) + [X, \theta([Y, Z])] - \theta([X, ([Y, Z])]) \\
&= [X, \nabla_Y(\theta(Z))] - [X, \nabla_Z(\theta(Y))] - \nabla_{[X, Y]}(\theta(Z)) + \\
&\quad + \theta([X, Y], Z) + \nabla_{[X, Z]}(\theta(Y)) + \theta([Y, [X, Z]]) - \\
&\quad - \nabla_Y([X, \theta(Z)]) + \nabla_Z([X, \theta(Y)]) - \theta([X, ([Y, Z])]) \\
&= [X, \nabla_Y(\theta(Z))] - [X, \nabla_Z(\theta(Y))] - \nabla_{[X, Y]}(\theta(Z)) + \\
&\quad + \nabla_{[X, Z]}(\theta(Y)) - \nabla_Y([X, \theta(Z)]) + \nabla_Z([X, \theta(Y)]) \\
&= (L_X\nabla)(Y, \theta(Z)) - (L_X\nabla)(Z, \theta(Y)).
\end{aligned}$$

Consequently,

$$(L_X(d_{\nabla}\theta))(Y, Z) = (d_{\nabla}(L_X\theta))(Y, Z) + (L_X\nabla)(Y, \theta(Z)) - (L_X\nabla)(Z, \theta(Y)).$$

This proves the theorem. $\square$

Corollary 3.11. *If X is a parallel vector field on $\mathbb{R}^n$ and $\nabla = D$ then we have*

$$L_X(d_D\theta) = d_D(L_X\theta), \tag{3.12}$$

for all $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$.

Proof. For all $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, $Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, applying Theorem 3.10, we obtain

$$(L_X(d_D\theta))(Y, Z) = (d_D(L_X\theta))(Y, Z) + (L_X\nabla)(Y, \theta(Z)) - (L_X\nabla)(Z, \theta(Y))$$

Since X is parallel vector field on $\mathbb{R}^n$, thus applying Theorem 3.5, we obtain

$$(L_X(d_{\nabla}\theta))(Y, Z) = (d_{\nabla}(L_X\theta))(Y, Z), \text{ for any } Y, Z \in \mathfrak{B}(\mathbb{R}^n).$$

Hence, $L_X(d_{\nabla}\theta) = d_{\nabla}(L_X\theta)$, for any $\theta \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$. $\square$

Proposition 3.12. *Suppose that $X, Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, $\omega \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$. Then we have*

$$(L_X (\nabla_Y \omega))(Z) - (\nabla_Y (L_X \omega))(Z) = (L_X \nabla_Y)(\omega(Z)) - \omega[[X, Y], Z]. \quad (3.13)$$

Proof. For all $X, Y, Z \in \mathfrak{B}(\mathbb{R}^n)$, $\omega \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, we have

$$\begin{aligned} (L_X (\nabla_Y \omega))(Z) &= [X, (\nabla_Y \omega)(Z)] - (\nabla_Y \omega)[X, Z] \\ &= [X, \nabla_Y(\omega(Z)) - \omega[Y, Z]] - \nabla_Y(\omega[X, Z]) + \omega[Y, [X, Z]] \\ &= [X, \nabla_Y(\omega(Z))] - [X, \omega[Y, Z]] - \nabla_Y(\omega[X, Z]) + \omega[Y, [X, Z]] \end{aligned} \quad (3.14)$$

$$\begin{aligned} (\nabla_Y (L_X \omega))(Z) &= \nabla_Y((L_X \omega)(Z)) - (L_X \omega)[Y, Z] \\ &= \nabla_Y([X, \omega(Z)] - \omega[X, Z]) - [X, \omega[Y, Z]] + \omega[X, [Y, Z]] \\ &= \nabla_Y[X, \omega(Z)] - \nabla_Y(\omega[X, Z]) - [X, \omega[Y, Z]] + \omega[X, [Y, Z]] \end{aligned} \quad (3.15)$$

From Equations (3.14, 3.15), we obtain

$$\begin{aligned} (L_X (\nabla_Y \omega))(Z) - (\nabla_Y (L_X \omega))(Z) &= [X, \nabla_Y(\omega(Z))] - \nabla_Y[X, \omega(Z)] + [Z, [X, Y]] \\ &= (L_X \nabla_Y)(\omega(Z)) - \omega[[X, Y], Z], \forall \omega \in \Omega^1(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n)). \end{aligned}$$

□

Acknowledgments. We would like to thank to Assoc. Prof. Dr. Nguyen Huu Quang, Assoc. Prof. Dr. Nguyen Huynh Phan, Dr. Kieu Phuong Chi for their encouragement and for reading the first draft of the paper.

References

- [1] R. Abraham, J.E. Marsden and T. Ratiu. *Manifolds, Tensor Analysis and Applications*. Springer, 2002.
- [2] T. Frankel. *The Geometry of Physics, An Introduction*. Cambridge, 2004.
- [3] Jean Gallier. *Notes on differential geometry and Lie groups*, Department of Computer and Information Science, University of Pennsylvania, Philadelphia, PA 19104, USA, 2012.
- [4] Katharina Habermann, Andreas Klein (2003), *Lie derivative of symplectic spinor fields, metaplectic representation, and quantization*. Math. Kolloq. 57, 711 .
- [5] Jung-Hwan Kwon and Young Jin Suh(1997), *Lie derivatives on Homogeneous real hypersurfaces of type a in complex space forms*. Bull. Korean Math. Soc. Vol 34, No. 3, pp. 459-468
- [6] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Interscience Publishers. New York - London. Vol. 1, 1963; Vol. 2, 1969.

- [7] Katharina Habermann, Andreas Klein, *Lie derivative of symplectic spinor fields, metaplectic representation, and quantization*, Rostock. Math. Kolloq. 57, 71-91, 2003.
- [8] B. L. Laptev, Lie differentiations, In: Progress in Science and Technology, Series on Algebra, Topology and Geometry [in Russian], All-Union Institute for Scientific and Technical Information, USSR Academy of Sciences, Moscow (1967), pp. 429465.12.7671[math-ph].
- [9] Dr.Felipe Leitner (2007), *Applications of Cartan and Tractor Calculus to Conformal and CR-Geometry*, Von der Fakultät Mathematik und Physik der Mathematics Universität Stuttgart als Habilitationsschrift genehmigte Abhandlung. Fachbereich Mathematik, Stuttgart.
- [10] J. S. B. Mitchell (2000), *Geometric shortest paths and network optimization*, in *Handbook of Computational Geometry*, J. R. Sack and J. Urrutia, eds, Elsevier Science B. V., pp. 633-701.
- [11] Paolo Piccione and Daniel V.Tausk(2006), *connections compatible with tensors. A characterization of left-invariant Levi-Civita connections on Lie groups*, Revista de la Unión Matemática Argentina. 1(47), 125-133.
- [12] Nguyen Huu Quang, Kieu Phuong Chi and Bui Cao Van, *The Lie derivative of currents on Lie group*, Lobachevskiy Journal of Mathematics. Vol. 33, No. 1, pp. 10 - 21, 2012.
- [13] Nguyen Huu Quang, Kieu Phuong Chi and Bui Cao Van, *On the Lie derivative of forms of bidegree*, Journal of Geometry and Physics, Submit 2014.
- [14] Nguyen Huu Quang and Bui Cao Van, *Some properties on the Lie derivative of currents and its applications*, Mathematics and Mechanics of Solids, Submit 2014.
- [15] Nguyen Huu Quang and Bui Cao Van, *The Lie derivative of normal connections*, Russian Mathematics, Submit 2014.
- [16] Nguyen Huu Quang, Bui Cao Van and Dang Thi Tuoi, *The Lie Derivative and the Exterior Derivative Connecting with Linear Connection on Algebra*, Applied Mathematical Sciences, Vol. 8, 2014, no. 125, 6223 - 6235.
- [17] R. P. Singh and S. D. Singh, *Lie Derivatives and Almost Analytic Vector Fields in a Generalised Structure Manifold*, Int. J. Contemp. Math. Sciences, 5(2), 81 - 90, 2010.
- [18] B.N. Shapukov, *Lie derivatives on fiber manifolds*, J. Math. Sci. 108(2), 211-231, 2002.
- [19] A. Ya. Sultanov, *Derivations of linear algebras and linear connections*, J. Math. Sci. 169(3), 362-412, 2010.
- [20] K. Yano, *The theory of Lie derivatives and its applications*, North-Holland Publishing Co., Amsterdam; P. Noordhoff Ltd., Groningen; Interscience Publishers Inc., New York, 1957.